2020 BMO

Problem 1. Let ABC be an acute triangle with $AB = AC$, let D be the midpoint of the side AC, and let γ be the circumcircle of the triangle ABD. The tangent of γ at A crosses the line BC at E. Let O be the circumcentre of the triangle ABE. Prove that the midpoint of the segment AO lies on γ.

Problem 2. Denote $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$ the set of all positive integers. Determine all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for each positive integer n,

i) $\sum\limits_{k=1}^{n} f(k)$ is a perfect square, and

ii) $f(n)$ divides n^3.

Problem 3. Let k be a positive integer. Determine the least integer n, with $n \geqslant k + 1$, for which the game below can be played indefinitely:

> Consider n boxes, labelled $b_1, b_2, \ldots, b_n$. For each index i, box b_i contains initially exactly i coins. At each step, the following three substeps are performed in order:
>
> **(1)** Choose $k + 1$ boxes;
>
> **(2)** Of these $k + 1$ boxes, choose k and remove at least half of the coins from each, and add to the remaining box, if labelled b_i, a number of i coins.
>
> **(3)** If one of the boxes is left empty, the game ends; otherwise, go to the next step.

Problem 4. Let $a_1 = 2$ and, for every positive integer n, let a_{n+1} be the smallest integer strictly greater than a_n that has more positive divisors than a_n has. Prove that $2a_{n+1} = 3a_n$ only for finitely many indices n.

Time: $4\,{}^1\!/_2$ *hours*
Each problem is worth 10 *marks*