THE MATHEMATICAL ASSOCIATION

National Committee for Mathematical Contests

British Mathematical Olympiad

18th March, 1982

Time allowed - 3½ hours

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order.

Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. PQRS is a quadrilateral of area A. O is a point inside it. Prove that if

$$2A = OP^2 + OQ^2 + OR^2 + OS^2,$$

then PQRS is a square and O is its centre.

2. A multiple of 17 when written in the scale of 2 contains exactly three digits 1. Prove that it contains at least six digits 0, and that if it contains exactly seven digits 0, then it is even.

3. If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$ and $n > 2$, prove

$$n(n+1)^a - n < s_n < n - (n-1)n^b$$

where a and b are given in terms of n by $an = 1$, $b(n-1) = -1$.

4. A sequence of real numbers $u_1, u_2, u_3 \ldots$ is given by u_1 and the recurrence relation $u_n^3 = u_{n-1} + \frac{15}{64}$, $n \geqslant 2$.

By considering the curve $x^3 = y + \frac{15}{64}$, or otherwise, describe with proof the behaviour of u_n as n tends to infinity.

5. A right circular cone stands on a horizontal base, radius r. Its vertex V is at a distance 1 from every point on the perimeter of the base. A plane section of the cone is an ellipse whose lowest point is L and whose highest point is H. On the curved surface of the cone, to one side of the plane VLH, two routes from L to H are marked. R_1 is along the semi-perimeter of the ellipse and R_2 is the route of shortest length.

Find the condition that R_1 and R_2 intersect between L and H.

6. Prove that the number of sequences $a_1 a_2 \ldots a_n$ with each of their n terms $a_i = 0$ or 1 and containing exactly m occurrences of 01 is $\binom{n+1}{2m+1}$.