United Kingdom Mathematics Trust

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British Mathematical Olympiad Round 1 2022

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2022 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. If a problem has two distinct logical parts, these are sometimes marked separately and the scores added, but one part is generally considered to be more challenging. For example, in Q4 we need to show (i) that Katy can always score at least 32, no matter what Alex does and (ii) that Alex can prevent Katy from scoring more than 32, no matter what strategy she follows. Here (i) requires more sophistication, and carries 6 of the 10 marks available. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

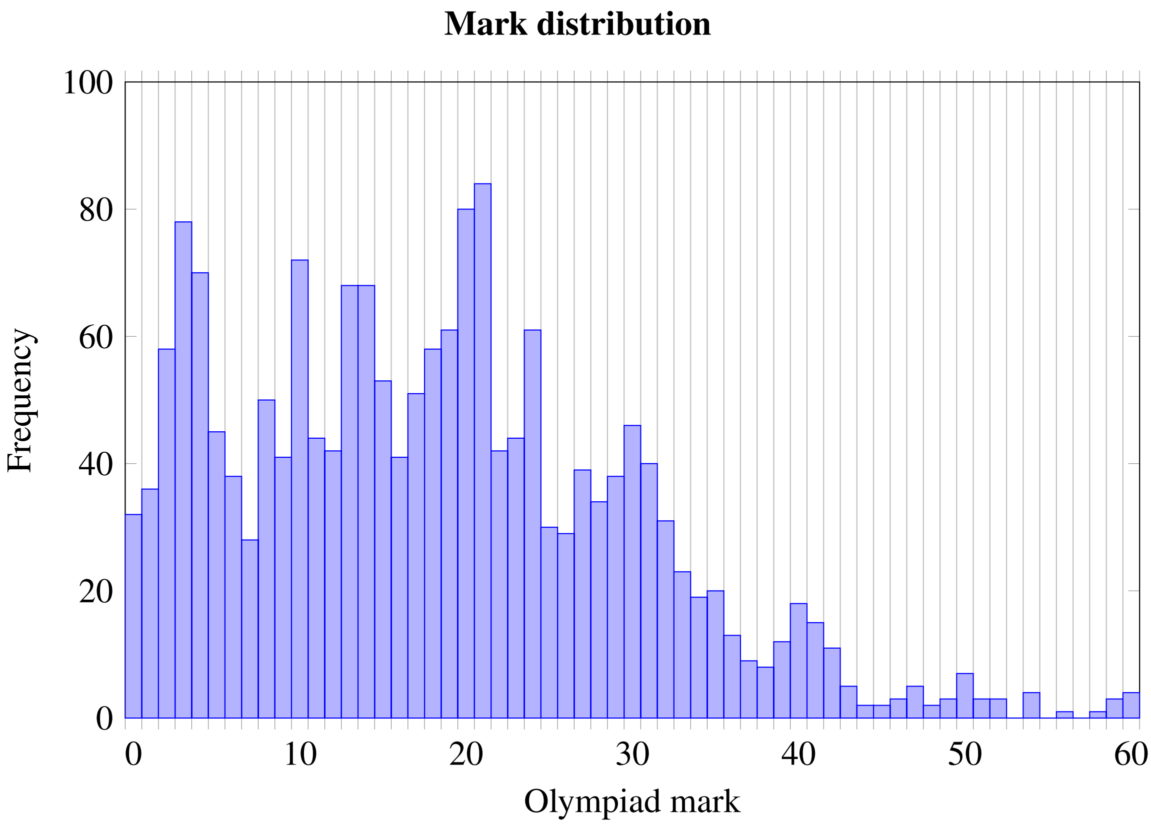
This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

Responses to this year's paper were mixed. An encouraging number of strong candidates made substantial progress on three, or even four of the problems, while only the very best were able to score highly on either of the last two questions. At the other end of the score distribution, while almost all candidates were able to engage sensibly with question 1, quite a number failed to adequately explain their reasoning. This led to a number of fairly low scores. It is hoped that candidates who obtained correct answers without scoring full marks (which was common on questions 1, 2 and 4) will not be too discouraged. Maths Olympiads aim to test two related but separate skills: solving problems and constructing mathematical arguments. Many of those with low total scores shone on the first aspect, but lacked the requisite experience to shine on the second. Candidates are reminded that it is good practice to work in rough before writing up their solutions, and also that it is vital to reread those solutions critically. Asking 'Is it clear *from what I have written* that there are no other solutions?' might have improved a number of responses to question 1, while in question 4 relevant questions would have included 'Is it clear that this strategy works no matter what the other player does?' and also 'Have I explained how Katy can always score at least 32 *and* how Alex can always prevent her from scoring more?'.

Taking care when putting pencil to paper and not rushing was, as ever, crucial in the geometry questions where large, accurate diagrams made it far easier to see what was going on (as well as being a great help to those marking the scripts). There were an impressive number of different approaches used in successful solutions to question 3, and it was clear that question 6 was intriguing, even to those who did not solve it.

The 2022 British Mathematical Olympiad Round 1 attracted 1909 entries. The scripts were marked in Cambridge (with some remote markers) from the 2nd to the 4th of December by a team of: Eszter Backhausz, Tibor Backhausz, Sam Bealing, Emily Beatty, Jonathan Beckett, Phil Beckett, James Bell, Robin Bhattacharyya, Andrew Carlotti, Helen Chen, Sam Childs, Andrea Chlebikova, James Cranch, Laura Daniels, Stephen Darby, Wendy Dersley, Joe Devine, Paul Fannon, Richard Freeland, Thomas Frith, Carol Gainlall, Chris Garton, Sarah Gleghorn, Aleksander Goodier, Amit Goyal, Ben Handley, Sarp Hangisi, Stuart Haring, Tom Hillman, Ian Jackson, Vesna Kadelburg, Hadi Khan, Kit Kilgour, Jeremy King, Patricia King, David Knipe, Larry Lau, Rhys Lewis, Samuel Liew, Aleksandar Lishkov, Thomas Lowe, Eleanor MacGillivray, Owen Mackenzie, Sam Maltby, Przemysław Mazur, Harry Metrebian, Kian Moshiri, Oliver Murray, Joseph Myers, Daniel Naylor, Martin Orr, Jenny Owladi, Preeyan Parmar, Dominic Rowland, Adrian Sanders, Alan Slomson, Geoff Smith, Anujan Sribavananthan, Stephen Tate, Velian Velikov, Tommy Walker Mackay, Zi Wang, Henry Wilson, Tianyiwa Xie, Harvey Yau, Dominic Yeo.



The thresholds for qualification for BMO2 were as follows:

Year 13: 38 marks or more.

Year 12: 36 marks or more.

Year 11: 34 marks or more.

Year 10 or below: 33 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 39 marks or more.

Distinction: 25 marks or more.

Merit: 12 marks or more.

Question 1

A road has houses numbered from 1 to n, where n is a three-digit number. Exactly $\frac{1}{k}$ of the numbers start with the digit 2, where k is a positive integer. Find the possible values of n.

SOLUTION

We consider the three cases $n < 200$, $200 \le n < 300$ and $300 \le n$ separately.

Case I

There are eleven houses among the first 99 whose numbers begin with a 2, house 2 and houses 20 to 29 inclusive.

Thus if $n < 200$, we have $\frac{1}{k} = \frac{11}{n}$. This shows that n is a multiple of 11. There are nine possibilities, namely $n = 110, 121, 132, 143, 154, 165, 176, 187, 198$.

Case II

If $200 \le n < 300$ we may write $n = 199 + x$ for some integer $1 \le x \le 100$.

We have that $\frac{1}{k} = \frac{11+x}{199+x}$ so $k = \frac{199+x}{11+x} = 1 + \frac{188}{11+x}$. This implies that $k - 1 = \frac{188}{11+x}$ so $11 + x$ must be a factor of 188 (since $k - 1$ is an integer).

Thus, we need to find factors of 188 which are between 12 and 111 inclusive. Since, $188 = 2^2 \times 47$ its factors are $1, 2, 4, 47, 94$ and 188. Of these we need only consider 47 and 94 which give $(x, n) = (36, 235)$ and $(83, 282)$ respectively.

Case III

There are 111 houses among the first 299 whose numbers begin with a 2, the eleven already counted and those numbered 200 to 299 inclusive.

Thus if $n \ge 300$, we have $\frac{1}{k} = \frac{111}{n}$. This shows that n is a multiple of 111. There are seven possibilities, namely $n = 333, 444, 555, 666, 777, 888, 999$.

ALTERNATIVE

The second, most interesting, case can also be tackled by bounding the possible values of k and then checking each in turn. Since $k - 1 = \frac{188}{11+x}$, and the left hand side decreases as x increases, we see that $\frac{188}{12} + 1 \ge k \ge \frac{188}{111} + 1$ or $3 \le k \le 16$. These fourteen values of k can be tested in turn to see that only two, $k = 3$ and $k = 5$, give rise to integer values of x and hence n. It is important with solutions of this kind to provide enough evidence of the checking to make it clear that the solutions have been found systematically, rather than by lucky guesswork.

MARKERS' COMMENTS

Many candidates found the solutions in the cases $100 \le n \le 199$ and $300 \le n \le 999$.

The difficult part of the problem was the case $200 \le n \le 299$; dealing carefully with this case, and finding at least some solutions in the other two ranges, was required for a script to be classified as 10-. A number of candidates found most, or even all of the solutions, but did not adequately justify why their lists were complete.

If candidates took the approach of an algebraic expression, justification for limiting the cases was required. A rearrangement to find $k - 1$ or $n - 188$ as a factor of 188 was sufficient, but a statement that $n - 188$ divides n implies that $n - 188$ divides 188 needed some justification.

If candidates took the approach of bounding k, justification of the bounds was necessary. Explicitly checking each value of k within those bounds or explanation why values did not lead to solutions was needed for full credit.

In both cases, candidates stating they have checked the cases was not enough on its own.

There were many misreads or miscounting numbers beginning in 2. Where these still led to problems of near identical difficulty, for example when candidates forgot the number '2' or the number '200', it was still possible to obtain nearly full marks.

Question 2

A sequence of positive integers a_n begins with $a_1 = a$ and $a_2 = b$ for positive integers a and b. Subsequent terms in the sequence satisfy the following two rules for all positive integers n:

$$a_{2n+1} = a_{2n}a_{2n-1}, \quad a_{2n+2} = a_{2n+1} + 4.$$

Exactly m of the numbers $a_1, a_2, a_3, \ldots, a_{2022}$ are square numbers. What is the maximum possible value of m? *Note that m depends on a and b, so the maximum is over all possible choices of a and b.*

SOLUTION

We begin by observing that no two positive square numbers differ by four. This can be seen by considering $a^2 - b^2 = (a - b)(a + b)$ which is at least 8 if it is even, or by noting that the gaps between the positive squares are the odd numbers starting with 3, no two of which sum to 4.

For $n \geq 1$ we have that a_{2n+1} and a_{2n+2} differ by 4, so at most one of them is a square.

For $n \geq 2$ we have that $a_{2n+2} = a_{2n}a_{2n-1} + 4$ which is $(a_{2n-1} + 2)^2$ and so always a square.

Thus $a_6, a_8, \ldots a_{2022}$ are all square and $a_5, a_7, \ldots a_{2021}$ are not, giving 1009 squares from a_5 to a_{2022}.

The sequence begins $a, b, ab, ab + 4$. The last two differ by 4 so are not both square which implies that at most three of the first four terms are squares. Moreover, if a and b are both square numbers, then ab will also be a square.

Thus there are at most 1012 squares among the first 2022 terms of the sequence. This is attained if (and only if) a and b are both squares.

MARKERS' COMMENTS

Many candidates studied the start of the sequence (either algebraically or using numerical examples), spotted the squares and thus obtained the correct answer of 1012. However, a number of these candidates did not adequately explain their reasoning.

A complete solution to this question generally consisted of three parts: (a) stating and proving that the even-numbered terms starting from a_6 are always square, (b) stating and proving that the odd-numbered terms starting from a_5 or a_7 are never square and finally, (c) carefully considering how many of a_1 to a_4 can be square.

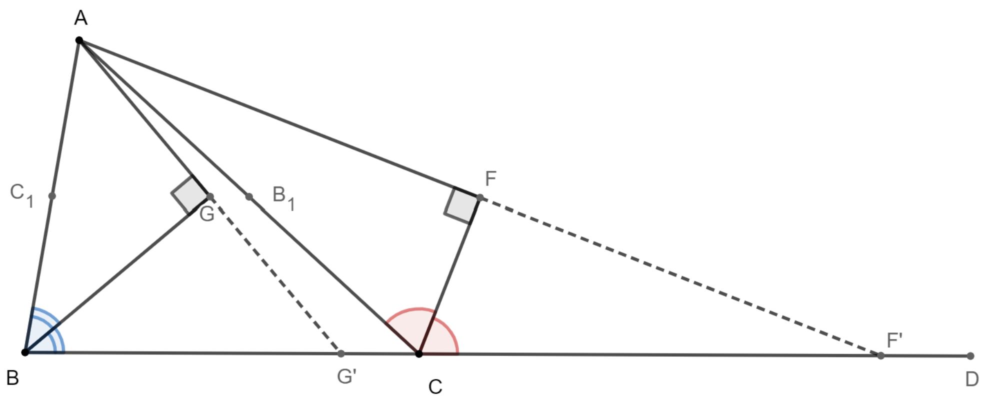
Many students made a strong start by proving (a), but ended up with low scores by not putting any note or justification of (b) on paper. Some may feel harshly penalised for omitting something they felt was obvious, but the omission leaves a gap in the logical structure of the argument. This is an essential idea that comes back in many problems, so it is worth remembering that if a question asks one to find *all* things with a certain property, one must

- find the things with said property (and show they indeed possess the property), and
- show that no other things have said property.

Question 3

In an acute, non-isosceles triangle ABC the midpoints of AC and AB are B_1 and C_1 respectively. A point D lies on BC with C between B and D. The point F is such that $\angle AFC$ is a right angle and $\angle DCF = \angle FCA$. The point G is such that $\angle AGB$ is a right angle and $\angle CBG = \angle GBA$. Prove that B_1, C_1, F and G are collinear.

SOLUTION



Extend AF and AG to meet BC at F' and G' respectively. As a consequence of ASA, we have the following congruent triangles:

$$\begin{cases} \angle FCA = \angle F'CF \\ CF = CF \\ \angle AFC = 90^\circ = \angle CFF' \end{cases} \Rightarrow \triangle AFC \cong \triangle F'FC \qquad \begin{cases} \angle GBA = \angle G'BG \\ BG = BG \\ \angle AGB = 90^\circ = \angle BGG' \end{cases} \Rightarrow \triangle AGB \cong \triangle G'GB$$

Therefore, $AF = FF'$ and $AG = GG'$ so F, G, B_1, C_1 are the midpoints of AF', AG', AC, AB respectively. This means:

$$\frac{AF}{AF'} = \frac{AG}{AG'} = \frac{AB_1}{AC} = \frac{AC_1}{AB} = \frac{1}{2}$$

So if we consider an enlargement with scale factor $\frac{1}{2}$ at A, then the line passing through B, G', C, F' maps to a line passing through C_1, G, B_1, F proving these four points are collinear.

ALTERNATIVE

Because B_1, C_1 are the midpoints of AC, AB respectively, we have $B_1C_1 \parallel BC$ so $\angle B_1C_1A = \angle B$. Also, $\angle AGB = 90^\circ$ so C_1 is the centre of circle AGB and we get:

$$\angle GC_1A = 2 \cdot \angle GBA = \angle GBA + \angle CBG = \angle B = \angle B_1C_1A$$

Thus G lies on B_1C_1.

Similarly, $\angle AFC = 90^\circ$ so B_1 is the centre of the circle AFC giving:

$$\angle FB_1A = 2\angle FCA = \angle FCA + \angle DCF = \angle DCA = 180^\circ - \angle C$$

And using $B_1C_1 \parallel BC$ we get:

$$\angle AB_1C_1 = \angle C \implies \angle AB_1C_1 + \angle FB_1A = 180^\circ$$

so F also lies on B_1C_1.

Alternative

(Sketch) We present the argument for G lying on B_1C_1. The argument for F is similar.

Let the B-internal angle bisector intersect B_1C_1 at $\tilde{G}$. We want to show G and $\tilde{G}$ are the same point. As G also lies on this angle bisector, it's sufficient to show $\angle A\tilde{G}B = 90°$.

To do this, observe that because $B_1C_1 \parallel BC$ and $B\tilde{G}$ is an angle bisector:

$$\angle \tilde{G}BC_1 = \angle CB\tilde{G} = \angle C_1\tilde{G}B \implies C_1\tilde{G} = C_1B = C_1A$$

Hence $\tilde{G}$ lies on the circle with diameter AB so $\angle A\tilde{G}B = 90°$ as desired.

Markers' comments

As with all geometry problems, a good place to start is to draw a large diagram with a compass and a ruler. Not only can this help give you ideas for how to solve the problem, it also makes it clearer to the marker where you have defined points (though you should always define them in your solution as well – not just mark them on the diagram). One thing to be careful of in this problem is not accidentally *assuming* that F, G lie on B_1C_1 at some point in your proof (particularly in a long angle chase). A helpful way to reduce the risk of doing this is to draw line B_1C_1 as a *dashed* line in your diagram.

We saw many successful solutions with some students coming up with approaches that were novel to the problem setters, which was great to see. Some students lost marks for not justifying key steps, for example: not relating B_1 being the centre of circle AFC to $\angle AFC = 90°$; or explaining why certain triangles were congruent; or justifying why $B_1C_1 \parallel BC$. There were also penalties for students who provided an argument for F lying on B_1C_1 and simply stated *the same argument also works for G* (or vice-versa). While the arguments are similar, in many approaches they weren't identical so at least some justification was required.

An approach that was employed to a varying degree of success was that of *phantom points* where we *define* a point $\tilde{G}$ to have certain properties and try to prove that in fact this is the same point as G. Properties of interest include:

(i) $\tilde{G}$ is such that $\angle CB\tilde{G} = \angle \tilde{G}BA$

(ii) $\angle A\tilde{G}B$ is a right-angle

(iii) $\tilde{G}$ lies on B_1C_1

The problem statement asks us to prove *(i)*, *(ii)* $\implies$ *(iii)* but we could equally try to show *(i)*, *(iii)* $\implies$ *(ii)* (which is shown in one of the example solutions). A common error in this approach was to, at some point in the proof, *assume* that $\tilde{G}$ had **all three** properties. Students can reduce the chance of making this mistake (and also help out the marker) by being clear at the start of their solution, what properties they are assuming and what they are trying to prove.

Question 4

Alex and Katy play a game on an 8×8 square grid made of 64 unit cells. They take it in turns to play, with Alex going first. On Alex's turn, he writes 'A' in an empty cell. On Katy's turn, she writes 'K' in two empty cells that share an edge. The game ends when one player cannot move. Katy's score is the number of Ks on the grid at the end of the game. What is the highest score Katy can be sure to get if she plays well, no matter what Alex does?

SOLUTION

Katy's maximum score is 32.

She can achieve this by dividing the board into 2×1 rectangles at the start of the game. On each of her turns she can place two Ks into an empty one of these rectangles. On Alex's turns he can reduce the number of empty 2×1 rectangles by at most 1. This means each player will have at least sixteen turns, giving Katy a final score of at least 32.

Alex can prevent Katy from scoring more than 32 as follows. He colours the board in the standard chessboard pattern and then promises to only ever place As on cells that are (say) black. This means that Alex and Katy each cover exactly one black cell each turn, so they can take at most 16 turns each, giving Katy a maximum possible score of 32.

REMARK

The delicate thing in questions of this type is to ensure that the strategies described for each player do not depend on the other playing following a particular 'sensible' strategy.

REMARK

There are other ways to describe good strategies for Katy. She might, for example, divide the board into sixteen 2×2 squares and whenever Alex places a first A in such a square, use her next move to place two Ks in that square.

MARKERS' COMMENTS

A complete solution here has two parts: (**A**) a strategy for Alex to stop Katy getting more than 32 cells, and (**K**) a strategy for Katy to get at least 32 cells. There were good attempts at both, though (**A**) was more popular. All solutions for (**A**) involved colouring the board in some way, but other strategies could produce a weaker bound than 32. All solutions for (**K**) involved dividing up the board into smaller regions in some way; again, other strategies could produce weaker bounds.

The key logical difficulty here is that (**A**) and (**K**) both need to work whatever the other player does. Lots of candidates found a strategy for Alex and then tried to show Katy could get 32 cells *when playing against that particular strategy.* The problem is that a cleverer Alex might find a better strategy. Other candidates made the same mistake the other way round, showing how Alex should play against a particular strategy by Katy.

A common small mistake of this type came in Katy's strategy based on 2×2 boards. If you're describing a general strategy, 'always follow Alex into the 2×2 board he took the first square of' is not enough, because Alex could return to a 2×2 board Katy already used. This looks like a bad move for Alex, so many candidates ignored it, but a general strategy must cover all cases.

Question 5

For each integer $n \geq 1$, let $f(n)$ be the number of lists of different positive integers starting with 1 and ending with n, in which each term except the last divides its successor. Prove that for each integer $N \geq 1$ there is an integer $n \geq 1$ such that N divides $f(n)$.

(So $f(1) = 1$, $f(2) = 1$ and $f(6) = 3$.)

Solution

We start by noting that $f(1) = 1$ since (1) is the only possible list.

For any given $n > 1$, we can count the number of allowed lists according to the penultimate number, d, in the list. Clearly d is a factor of n and the number of lists of the form $(\ldots, d, n)$ is $f(d)$.

Thus

$$f(n) = \sum_{d|n,\ \ d \neq n} f(d) \tag{†}$$

For example, $f(2) = f(1) = 1$, $f(4) = f(2) + f(1) = 2$, $f(8) = f(4) + f(2) + f(1) = 4$.

Further experimentation with small cases leads to the conjecture that $f(p^m) = 2^{m-1}$ for any prime p and positive integer k.

This claim can be proved by induction on m.

In clearly holds for $m = 1$ as $f(p) = 1$.

If we assume it holds for all integers m up to some integer k we may consider $f(p^{k+1})$.

$$\begin{aligned} f(p^{k+1}) &= f(p^k) + f(p^{k-1}) + \cdots + f(p) + f(1) \\ &= (2^{k-1} + 2^{k-2} + \cdots + 1) + 1 \\ &= (2^k - 1) + 1 \end{aligned}$$

Here the first line uses the recurrence (†), the second uses the inductive hypothesis and the third uses the sum of a geometric progression.

This is enough to prove the claim for $m = k + 1$ and thus, by induction, for all $m \geq 1$.

Next we claim that $f(p^m q) = (m + 2) \times 2^{m-1}$ for different primes p, q with $m \geq 1$. Again we proceed by induction on m.

If $m = 1$ we must check that $f(pq) = f(p) + f(q) + f(1) = 1 + 1 + 1 = (1 + 2) \times 2^0$ as required.

Now we assume the claims for all $m \leq k$ and consider $f(p^{k+1}q)$.

The key thing note is that the proper factors of $p^{k+1}q$ are $p^k q, p^{k+1}$ and a collection of other factors which are precisely the proper factors of $p^k q$. Thus

$$\begin{aligned} f(p^{k+1}q) &= f(p^k q) + f(p^{k+1}) + f(p^k q) \\ &= (k + 2)2^{k-1} + 2^k + (k + 2)2^{k-1} \\ &= (k + 3)2^k \end{aligned}$$

Here the first line uses (†) (rather cunningly) and the second uses the inductive hypothesis and our first claim.

Now for a given N we may choose any pair of primes p, q and set $n = p^{N-2}q$.

ALTERNATIVE

Given a legal sequence $1 = a_0, a_1, \ldots, a_k = n$ where $a_i | a_{i+1}$ and $a_i < a_{i+1}$ we can let $d_i = a_i/a_{i-1}$ for $1 \leq i \leq k$ to obtain the sequence $d_1, d_2, \ldots, d_k$ which in an ordered factorisation of n.

We can show that $n = p^m q$ has $(m+2) \times 2^{m-1}$ ordered factorisations by counting them directly.

We start by writing $n = p \times p \times \cdots \times q \times \cdots \times p$ and change some of the $\times$ signs into commas to obtain a list $d_1, d_2, \ldots$. To avoid any over counting we can insist that the q either appears at the far right of the list of or directly before a comma. This gives two cases. If we start with $n = p \times p \times \cdots \times q$ we may convert any subset of the m multiplication signs into commas. This gives 2^m options. If, on the other hand, q is not the last prime in our initial factorisation of n must choose its position in one of m ways and replace the $\times$ directly after it with a comma. We now choose any subset of the remaining $m-1$ multiplication signs to convert to commas in one of 2^{m-1} ways.

This gives a final count of $2^m + m2^{m-1} = (m+2)2^{m-1}$ as required.

ALTERNATIVE

We can also establish that $n = p^m q$ has $(m+2)2^{m-1}$ ordered factorisations by counting these factorisations according to the number of factors (that is, according to length of the list $d_1, d_2, \ldots$).

To obtain a factorisation with k factors we either divide the m copies of p into k blocks and add the q to one of them, or divide the copies of p into $k-1$ blocks and add the q as a block on its own.

The first option can be done in $k\binom{m-1}{k-1}$ ways since we must place $k-1$ 'dividers' into the $m-1$ spaces between the ps. The second option can be done in $k\binom{m-1}{k-2}$ ways since, having split the ps into $k-1$ blocks, there are k places to insert the single q.

Now the final count, $f(p^m q)$, can be simplified using standard combinatorial identities as follows.

$$\begin{aligned}
f(p^m q) &= \sum_k k\left(\binom{m-1}{k-1} + \binom{m-1}{k-2}\right) \\
&= \sum_k k\binom{m}{k-1} \\
&= \sum_k (k-1)\binom{m}{k-1} + \sum_k \binom{m}{k-1} \\
&= m2^{m-1} + 2^m
\end{aligned}$$

as required.

Markers' comments

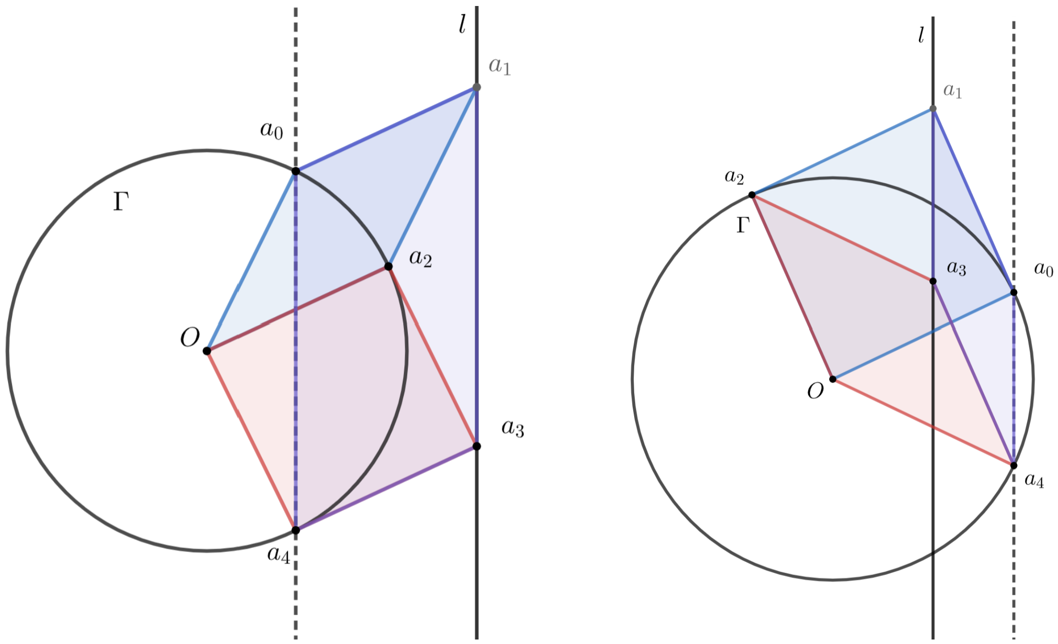
A Maths Olympiad coach once said "all construction problems are trivial". In some ways this can be true, in that the contestant has free choice to come up with any workable construction, but this freedom of choice frequently leads to candidates pursuing unhelpful directions. So it turned out to be with question 5 this year, where large numbers of candidates tried multiplying distinct primes together, and relatively few hit on the workable idea of $p^k q$. Unfortunately for the majority, the sequence of values that arises from using k distinct primes (called the Ordered Bell Numbers) cannot lead to a solution as they are all odd, amongst other issues. Many students only considered lists of length 3, or only considered the prime factors of n, both of which made the distinct prime approach appear to work, but unfortunately these solutions broke down when the other factors were included.

For those who did pursue the $p^k q$ approach, there were a wide variety of ways of finding the recurrence $f(p^k q) = 2f(p^{k-1}q) + 2^{k-1}$, with some being more number theoretic and others more combinatorial. For those who got to this point, the final hurdle that most commonly caused difficulties was noticing that the expression $(k+2)2^{k-1}$ isn't always a multiple of $k+2$, in particular when $k = 0$. This is not a big problem, as the $n = 2$ case is easily dealt with, but it did need a mention.

Question 6

A circle Γ has radius 1. A line l is such that the perpendicular distance from l to the centre of Γ is strictly between 0 and 2. A frog chooses a point on Γ whose perpendicular distance from l is less than 1 and sits on that point. It then performs a sequence of jumps. Each jump has length 1 and if a jump starts on Γ it must end on l and vice versa. Prove that after some finite number of jumps the frog returns to a point it has been on before.

Solution



Let O be the centre of Γ and let the points sat on by the frog be $a_0, a_1, a_2, \ldots$ in that order.

If the Frog ever jumps straight back to the point it just left we are done, so from now on we may assume this is not the case. That is, that $a_{i+2} \neq a_i$.

Since all the jumps have length 1, and a_0, a_2 are on Γ, we see that $Oa_0a_1a_2$ is a rhombus, so a_0a_1 is parallel to a_20.

Similarly $Oa_2a_3a_4$ is a rhombus, so a_3a_4 is parallel to a_20.

This means that $a_0a_1a_3a_4$ is a parallelogram since $|a_0a_1| = |a_3a_4| = 1$.

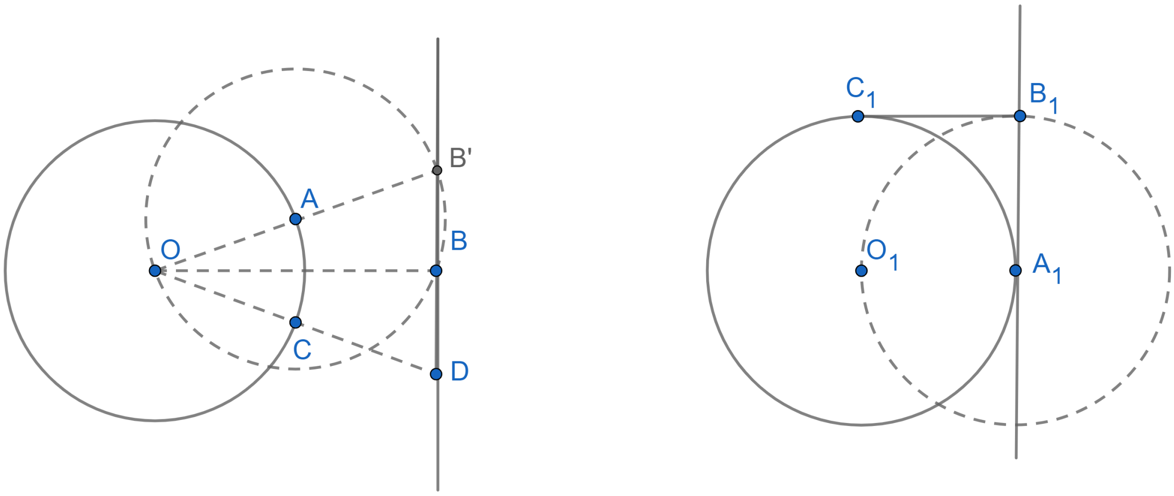
Now a_1 and a_3 are both on l, so the line a_0a_4 is parallel to l.

Adding 4 to all the indices and repeating the above argument, we see that the line a_4a_8 is also parallel to l. However, the unique line through a_4 parallel to l intersects Γ in (at most) one other point, so $a_0 = a_8$. Thus the frog revisits a previously visited point after at most eight jumps.

There is another way to clinch the argument. Once you establish that the vectors $\overline{a_0a_1}$, $\overline{Oa_2}$ and $\overline{a_4a_3}$ are equal, translation by this vector carries triangle $\triangle a_0Oa_4$ to the congruent triangle $\triangle a_1a_2a_3$ and so the line a_0a_4 is parallel to the line a_1a_3 which is l.

Remark

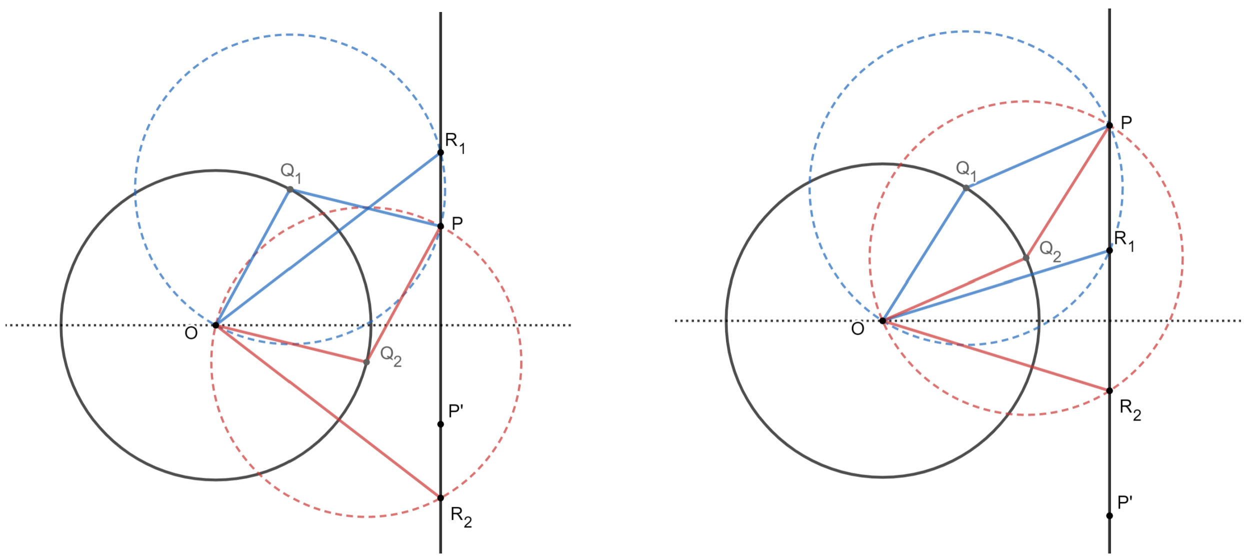
It is possible when the frog jumps from a_i to a_{i+1}, it finds that there is only one point which is a candidate for a_{i+2}. (This occurs when the circle radius 1 centre a_{i+1} is tangent to the other landing zone at a_{i+2}, and so either Oa_ia_{i+1} are collinear or a_ia_{i+1} is perpendicular to l). This case is covered by the remark that if $a_{i+2} = a_i$ then we are done. The figure shows examples of such configurations. The left diagram illustrates infinitely many configurations, as you vary the distance between O and the line l.



Remark

There are two possible versions of the diagram, one where l intersects Γ and one where it does not, but the argument given works without adjustment in both.

Alternative



Let O be the centre of Γ, let l_2 be the line through O perpendicular to l and let P be a point on l. From P, the frog can jump to two points Q_1, Q_2 on Γ. From Q_i the frog can jump to a single point $R_i \neq P$.

We have that P, R_i, O lie on a circle with centre Q_i. We claim that OR_1R_2 is isosceles with apex O, which implies that R_1 and R_2 are reflections of each other in l_2. To prove the claim we note that in the rhombus OQ_1PQ_2 the angles at Q_1 and Q_2 are equal. We then use the fact that the angle at the centre is double the angle at the circumference. There are essentially two cases depending on whether or not P lies between R_1 and R_2. (These can be dealt with simultaneously using directed angles.)

Now let P' be the reflection of P in M. By symmetry of l and Γ in l_2, we see that if the frog starts at P', after two moves it is at one of $\{P', R_1, R_2\}$.

Thus if $S = \{P, P', R_1, R_2\}$ then if the frog starts at a point in S, after two jumps it is still at a point in S. Thus after at most 8 jumps, it returns to a point it has already visited by the pigeonhole principle.

ALTERNATIVE

One can use isogonal conjugacy to solve this problem. The circumcentre of triangle a_1Oa_3 is a_2. Let this triangle have orthocentre H, the isogonal conjugate of a_2. Thus OH is perpendicular to l and passes through O. The angles around O and the fact that $|Oa_0| = |Oa_4|$ force a_0 and a_4 to be mutual reflections in the line through O which is perpendicular to l. Therefore $a_0 = a_8$ by restarting the frog at a_4.

This is an echo of the classical explanation of isogonal conjugacy: if P is a point in the plane of triangle $\triangle ABC$, then its isogonal conjugate P^* is on the perpendicular bisector of the line segment joining the reflections of P of any pair of sides of $\triangle ABC$. Thus P^* is the circumcentre of the triangle with vertices which are the reflections of P in the three sides of $\triangle ABC$.

ALTERNATIVE

Other solutions are possible. You can establish the parallelism of the two parallel lines by angle chasing arguments, often involving the angles around O. There are also arguments where you set up a geometric configuration with reflection symmetry about a line through the centre of Γ which is perpendicular to l, and then argue that it is a legal path for the frog and so that is where the frog must go. Attempts using this method should be scrutinized particularly carefully, since it is very easy to make unjustified assumptions using this approach.

MARKERS' COMMENTS

The vast majority of candidates did not submit serious attempts at this question, and presumably even well prepared candidates would not have seen a question quite like this one before. Here having the patience to draw a number of accurate figures, and then to invest some time in staring at them, is probably the best way to try and spot what is going on. Noticing the relevant parallel lines, rhombuses and symmetry is certainly the harder part of the problem; putting together the details at the end is straightforward in comparison.