FURTHER INTERNATIONAL SELECTION TEST

May 5th, 1975 $3\frac{1}{2}$ hours

1. In this question a "real function" means a function f such that $f(x)$ exists and is real for all real numbers x.

(i) Show that there is only one value of the constant b for which a real function f exists with the property that, for all real x and y,

$$f(x-y) = f(x) - f(y) + bxy.$$

(ii) A real function f has the property that for all real x and y

$$f(x+y) = f(x) + f(y) + cxy$$

where c is a constant.

(a) Prove that if f is continuous at $x = 0$ then it is continuous everywhere.

(b) Prove that if f is differentiable at $x = 0$ then it is differentiable everywhere, and find the most general f in this case.

Give candidates [illegible] (a) & (b) for (i) as well as (ii)

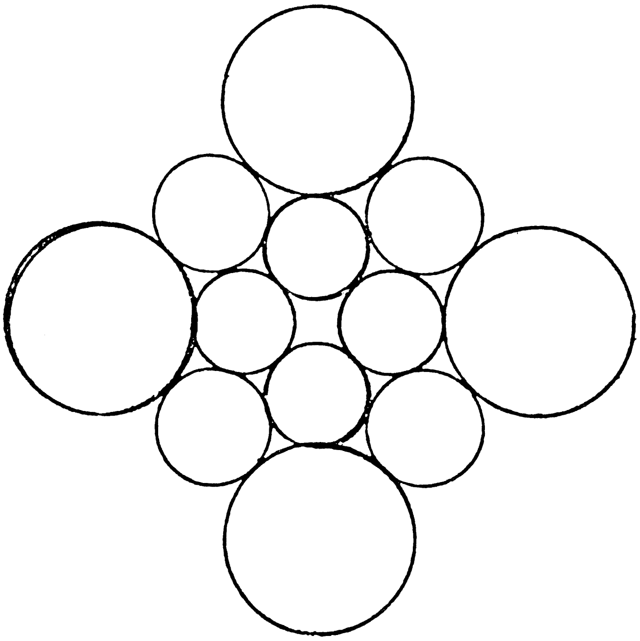
2. Prove that every positive integer which is not a member of the infinite set below is equal to the sum of two or more members of the set:

$$3,\ -2,\ 2^2.3,\ -2^3,\ \ldots,\ 2^{2k}.3,\ -2^{2k+1},\ \ldots\ .$$

3. There are n countries taking part in an international mathematical competition, with two contestants from each country. The competition is held in two rooms, A and B. At the start of the competition the $2n$ contestants form a queue, in any order. The contestant at the head of the queue enters room A. Each subsequent contestant goes first to the door of the room which his immediate predecessor in the queue entered, and looks in. If his fellow-countryman is not already in the room he enters it; otherwise he enters the other room. (So competitors from the same country are separated.) If all orders of queueing are equally likely determine with proof the probability that room A is filled with n contestants before room B.

Turn over

4.



The diagram illustrates a configuration of 12 circles. The set S of 12 circles contains three subsets S_3, S_4, S_5 each having 4 circles and such that each of the 4 circles of S_r touches r circles of S.

Prove that such a configuration of 12 circles exists on the surface of a sphere with all the 12 circles having equal radii.