NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1977

May 11th 1977 - 3½ hours

1. $X_0, X_1, \dots X_k$ are given points of the interval $[-1,1]$ such that

$$X_0 = -1,\ X_k = 1,\quad 0 < X_i - X_{i-1} \leqslant \tfrac{1}{2}\quad (1 \leqslant i \leqslant k)$$

The quadratic function f, of the form

$$f(x) = ax^2 + bx + c,$$

where a,b,c are real constants, satisfies the condition

$$|f(x_i)| \leqslant 1 \qquad (0 \leqslant i \leqslant k)$$

Prove that

$$|f(x)| \leqslant \frac{17}{15}$$

for all x in $[-1,1]$. Show by means of an example that this proposition becomes false if $\frac{17}{15}$ is replaced by any smaller number.

2. A pyramid is formed by joining the vertices of a plane quadrilateral ABCD, the "base", to a point V outside its plane. It is found that the inscribed circles of each pair of adjacent triangular faces touch each other. Prove that the points of contact of the inscribed circles with the base of the pyramid lie on a circle.

3. EITHER (a) Prove that if n is any given integer then the equation

$$10xy + 17yz + 27zx = n$$

has a solution in integers x,y,z.

OR (b) Prove that in the arithmetic progression

$$a,\ a+d,\ a+2d,\ \dots,\ a+nd,\ \dots$$

where a,d are positive integers, there exists an infinite set of terms having the same prime divisors.

4. Prove that for each integer $n > 1$ it is possible to construct a necklace having $2n^2$ beads in all, these being of 2n different colours, in such a way that for each pair of different colours there is at least one pair of adjacent beads of these two colours. Is it possible to do the same using $2n^2-1$ beads in all? Give a reason for your answer. (A "necklace" is a circular arrangement of beads, with no fastener intervening; an ample supply of beads of all the colours is assumed to be available).