## NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

## Further International Selection Test

Friday 5th February 1988

Time allowed - 3½ hours

### PLEASE READ THESE INSTRUCTIONS CAREFULLY

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet. On every sheet of working write your name and initials, and the number of the question.

There is no restriction on the number of questions which may be attempted.

1. ABC is an equilateral triangle. The circle $\Gamma_1$ has centre A and radius AB. $\Gamma_2$ is the circle on AB as diameter. A circle with centre P on AC touches $\Gamma_1$ internally at C and $\Gamma_2$ externally at Q. Show that AP/AC = 4/5 and calculate the ratio AQ/AC.

2. Prove that the number of ways of arranging 2n distinguishable objects in n pairs is

$$1 . 3 . 5 . \dots . (2n-1)$$

if the order of the pairs and the order of the objects within each pair are immaterial. Eg for 4 objects a, b, c, d the 3 pairings are ab, cd; ac, bd; ad, bc.

A party of 10 people consisting of 5 married couples is to be split into 5 pairs. A pair may consist of two men, two women or a man and a woman, but must not be a married couple. In how many ways can this arrangement be made if order, as above, is immaterial? Explain your reasoning carefully.

TURN OVER

3. The numbers $a, b, c, x, y, z$ satisfy the equations

$$x^2 - y^2 - z^2 = 2ayz$$
$$-x^2 + y^2 - z^2 = 2bzx$$
$$-x^2 - y^2 + z^2 = 2cxy$$

and also $$xyz \neq 0.$$

By using the first two equations express $z$ in terms of $a, b, x, y$. Prove that

$$x^2(1-b^2) = y^2(1-a^2) = xy(ab-c)$$

and hence find the value of $a^2 + b^2 + c^2 - 2abc$ (independently of $x, y, z$).

4. Find, with proof, all solutions of

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$$

where $x, y, z$ are positive integers.

5. $L$ and $M$ are two skew lines in space, ie they neither meet nor are parallel. $A, B$ are the points on $L, M$ respectively such that $AB$ is perpendicular to both $L$ and $M$. Points $P$ on $L$, $Q$ on $M$ vary so that

$$P \neq A, \quad Q \neq B, \quad PQ \text{ is of constant length.}$$

Show that the centre of the sphere through $A, B, P, Q$ lies on a fixed circle with centre the midpoint of $AB$.

6. Prove that if $a_1, b_1, c_1$ and $a_2, b_2, c_2$ are the lengths of the sides of two triangles (in some unit of measurement) then

$$a = \sqrt{(a_1{}^2 + a_2{}^2)}, \quad b = \sqrt{(b_1{}^2 + b_2{}^2)}, \quad c = \sqrt{(c_1{}^2 + c_2{}^2)}$$

are also the lengths of the sides of some triangle.