From Classroom to Contest

UKMT
2016 IMO Celebration

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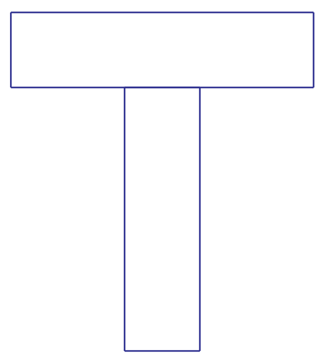
Contents

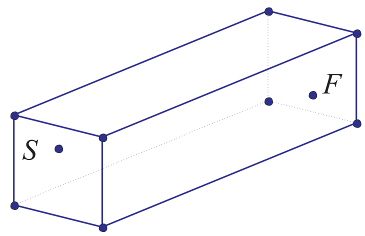
1 Discussed problems

1. (EMCC 2012) Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

2. (EMCC 2012) In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).

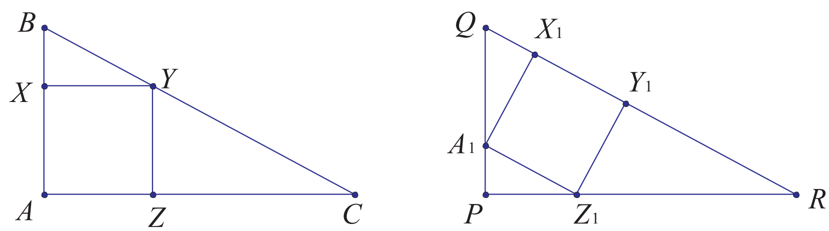


3. In the diagram shown below, a spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day, the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider crawl?



4. (Based on MPG 2016) In convex equilateral heptagon $HEXAGON$, $\angle H = \angle A = 168^\circ$ and $\angle E = \angle X = 108^\circ$. Find the respective degree measures of $\angle G, \angle O, \angle N$.

5. There are two common ways to put a square inside a right triangle. If the lengths of the legs are 5 and 12, what is the side length of each square?



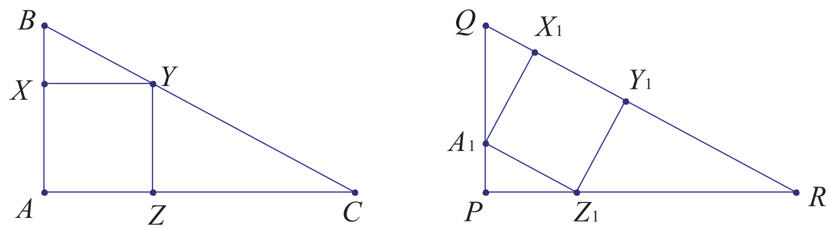
For an arbitrary right triangle, which way leads to the bigger square?

6. (RMM 2013, by Nikolai Beluhov from Bulgaria) Let $\mathcal{P}$ and $\mathcal{Q}$ be two convex quadrilateral regions in the plane which have a common point O (regions contain their boundary). Suppose that for every line ℓ through O the segment of intersection of ℓ and $\mathcal{P}$ is longer than the segment of intersection of ℓ and $\mathcal{Q}$. Is it possible that the ratio of the area of $\mathcal{Q}$ to the area of $\mathcal{P}$ is greater than 1.9?

2 Problems with hints given

7. (EMCC 2014, by Zhuoqun Alex Song) We say a polynomial P in x and y is n-good if $P(x, y) = 0$ for all integers x and y, with $x \neq y$, between 1 and n, inclusive. Determine the minimal degree of a nonzero 4-good polynomial.

8. (AIME 1987) Squares S_1 and S_2 are inscribed in the two congruent right triangle ABC and PQR, as shown below. Find $AB + AC$ if $[AXYZ] = 441$ and $[A_1X_1Y_1Z_1] = 440$.



9. (AIME 2009 packet, By Richard Parris) Let C be a corner of a $3 \times 1 \times 1$ rectangular solid $\mathcal{R}$, and let P be the non-interior point of $\mathcal{R}$ whose distance d from C, when measured along the surface of $\mathcal{R}$, is maximal. Given that $d^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.

10. (IMO 2006, by Zuming Feng) Determine all pairs (x, y) of integers such that
$$1 + 2^x + 2^{2x+1} = y^2.$$

3 You are on your own

11. In the city of Exercise, the capital of the Fat Republic, 2016 families who are unhappy with their current apartment are part of a project to exchange apartments. Each family is allowed to exchange apartments with another family once each day; only pairwise apartment exchanges are allowed. (Multiple pairs of families can exchange apartments in any given day.) It is known that there is a way of pairing families with apartments so that each family is in an apartment that it likes. What is the minimum number of days needed to guarantee that the families can switch to this arrangement, regardless of what that arrangement is?

12. (ELMO 2006, by Baohua Zhan) Numbers from 1 to n^2 is put into an $n \times n$ square array with no repetitions. A path is a series of movements

from the bottom left to the top right corner of the square, moving only up or to the right (all paths cover exactly $2n-1$ squares). The weight of a path is the sum of all numbers covered by the path. For any numbered square we can find a path with largest weight and a path with the smallest weight. Among all possible ways to number a square, what is the smallest possible difference between the weights of these two paths?

Folding open the box, for instance, hinging it along edge GH as shown below, will do the job. We only need to find the shortest path on a plane, which is easy to do. There are 6 such routes: heading for the midpoint of a nonadjacent edge on each face containing F.