Romanian Master in Mathematics
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Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

Problem 2. Prove that any bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$ where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

Problem 3. Given a positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n \geq 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor.$$

Here $\lfloor x \rfloor$ denotes the integer part of the real number x. This is the largest integer not greater than x.

Problem 4. Consider a square of side length a positive integer n. Suppose that there are $(n+1)^2$ points in the interior (i.e. strictly inside) the square. Show that three of these points define a (possibly degenerate) triangle of area at most $\frac{1}{2}$. *A triangle is degenerate if its vertices are collinear.*

Each problem is worth 7 points.

Time allowed is 5 hours.

The language of this version of the paper is not quite the same as the original. It has been modified to address questions of clarification posed to the jury by the contestants.