THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 2: SATURDAY, FEBRUARY 27, 2010, BUCHAREST

Language: English

Problem 4. Determine whether there exist a polynomial $f(x_1, x_2)$ in two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying all the following conditions:

(i) A is an integer point (i.e., a_1 and a_2 are integers);
(ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;
(iii) $f(n_1, n_2) > f(a_1, a_2)$, for all integer points (n_1, n_2) in the plane other than A;
(iv) $f(x_1, x_2) > f(b_1, b_2)$, for all points (x_1, x_2) in the plane other than B.

Problem 5. Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is *connected* if for every pair of points $R, S \in K$, there exist a positive integer ℓ and a sequence $R = T_0, T_1, \ldots, T_\ell = S$ of points in K, where each T_i is distance 1 away from T_{i+1}. For such a set K, we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n+1$ points with integer coordinates in the plane?

Problem 6. Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \ldots$ by $f^0(\mathbb{Q}) = \mathbb{Q}$ and $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S, we write $f(S)$ for the set $\{f(x) \mid x \in S\}$.)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Each of the three problems is worth 7 points.
Time allowed: $4\frac{1}{2}$ hours.