Mathematical Olympiad for Girls

Thursday 26th September 2013

Organised by the United Kingdom Mathematics Trust

Instructions

1. Do not turn over until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Full written solutions — not just answers — are required, with complete proofs of any assertions you may make.

   Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
4. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
5. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
6. Some questions have two parts. Part (a) introduces results or ideas useful in solving part (b).
7. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
8. Start each question on a fresh sheet of paper. Write on one side of the paper only.

   On each sheet of working write the number of the question in the top left-hand corner and your name, initials and school in the top right-hand corner.
9. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
10. Staple all the pages neatly together in the top left hand corner.
11. To accommodate candidates sitting in other timezones, please do not discuss the paper on the internet until 08:00 BST on Friday 27th September.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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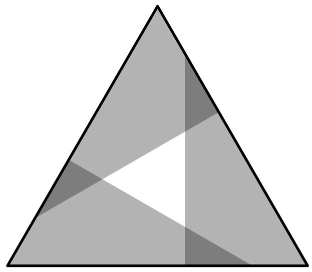
1. The diagram shows three identical overlapping *right-angled* triangles, made of coloured glass, placed inside an equilateral triangle, one in each corner. The total area covered twice (dark grey) is equal to the area left uncovered (white).

What fraction of the area of the equilateral triangle does one glass triangle cover?



2. *In triangle ABC, the median from A is the line AM, where M is the midpoint of the side BC. In any triangle, the three medians intersect at the point called the centroid, which divides each median in the ratio* $2 : 1$.

In the convex quadrilateral $ABCD$, the points A', B', C' and D' are the centroids of the triangles BCD, CDA, DAB and ABC, respectively.

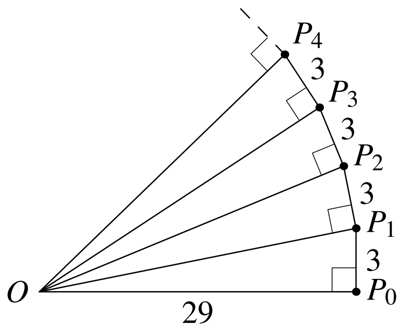
(a) By considering the triangle MCD, where M is the midpoint of AB, prove that $C'D'$ is parallel to DC and that $C'D' = \frac{1}{3}DC$.

(b) Prove that the quadrilaterals $ABCD$ and $A'B'C'D'$ are similar.

3. (a) Find all positive integers a and b for which $a^2 - b^2 = 18$.

(b) The diagram shows a sequence of points P_0, P_1, P_2, P_3, P_4, $\ldots$, which spirals out around the point O. For any point P in the sequence, the line segment joining P to the next point is perpendicular to OP and has length 3. The distance from P_0 to O is 29.

What is the next value of n for which the distance from P_n to O is an integer?



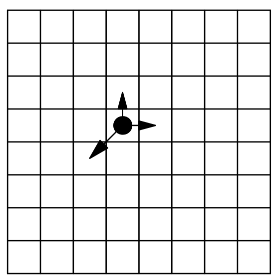
4. (a) An ant can move from any square on an 8×8 chessboard to an adjacent square. (Two squares are adjacent if they share a side).

The ant starts in the top left corner and visits each square exactly once. Prove that it is impossible for the ant to finish in the bottom right corner.

[*You may find it helpful to consider the chessboard colouring.*]

(b) A ladybird can move one square up, one square to the right, or one square diagonally down and left, as shown in the diagram, and cannot leave the board.

Is it possible for the ladybird to start in the bottom left corner of an 8×8 board, visit every square exactly once, and return to the bottom left corner?



5. (a) Find an integer solution of the equation $x^3 + 6x - 20 = 0$ and prove that the equation has no other real solutions.

(b) Let x be $\sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}$.

Prove that x is equal to 2.