Mathematical Olympiad for Girls

Tuesday 23rd September 2014

Organised by the United Kingdom Mathematics Trust

Instructions

1. Do not turn over until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. Full marks will be awarded for written solutions — not just answers — with complete proofs of any assertions you may make.

   Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.

   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Some questions have two parts. Part (a) introduces results or ideas useful in solving part (b).
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on a fresh sheet of paper. Write on one side of the paper only.

   On each sheet of working write the number of the question in the top left-hand corner and your name, initials and school in the top right-hand corner.
8. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
9. Staple all the pages neatly together in the top left hand corner.
10. To accommodate candidates sitting in other timezones, please do not discuss the paper on the internet until 08:00 BST on Wednesday 24th September.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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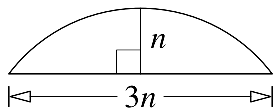
1. A chord of a circle has length $3n$, where n is a positive integer. The segment cut off by the chord has height n, as shown.



What is the smallest value of n for which the radius of the circle is also a positive integer?

2. (a) Some strings of three letters have the property that all three letters are the same; for example, LLL is such a string.

How many strings of three letters do not have all three letters the same?

(b) Call a number 'hexed' when it has a recurring decimal form in which both the following conditions are true.

(i) The shortest recurring block has length six.
(ii) The shortest recurring block starts immediately after the decimal point.

For example, $987.\dot{1}23\,45\dot{6}$ is a hexed number (the dots indicate that $123\,456$ is a recurring block).

How many hexed numbers are there between 0 and 1?

3. A large whiteboard has 2014 + signs and 2015 − signs written on it. You are allowed to delete two of the symbols and replace them according to the following two rules.

(i) If the two deleted symbols are the same, then replace them by +.
(ii) If the two deleted symbols are different, then replace them by −.

You repeat this until there is only one symbol left. Which symbol is it?

4. (a) In the quadrilateral $ABCD$, the sides AB and DC are parallel, and the diagonal BD bisects angle ABC. Let X be the point of intersection of the diagonals AC and BD.

Prove that $\dfrac{AX}{XC} = \dfrac{AB}{BC}$.

(b) In triangle PQR, the lengths of all three sides are positive integers. The point M lies on the side QR so that PM is the internal bisector of the angle QPR. Also, $QM = 2$ and $MR = 3$.

What are the possible lengths of the sides of the triangle PQR?

5. The AM-GM inequality states that, for positive real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Moreover, equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

(a) Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

for all positive real numbers a, b and c, and determine when equality holds.

(b) Find the minimum value of

$$\frac{a^2}{b} + \frac{b}{c^2} + \frac{c}{a}$$

where a, b and c are positive real numbers.