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Mathematical Olympiad for Girls 2015

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has some sort of overall strategy or not. An answer which is essentially a solution, but might contain either errors of calculation, flaws in logic, omission of cases or technical faults, will be marked on a '10 minus' basis. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which shows no sign of being a genuine solution is marked on a '0 plus' basis; up to 3 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

We were impressed with the high standard of the competition this year. It was particularly pleasing to see so many candidates having the confidence to engage with later questions: a vast majority scored marks on Question 4 and nearly half did so on Question 5.

The quality of explanations was also improved compared to previous years, particularly in geometry. Most candidates clearly understand the importance of justifying their reasoning rather than giving a single numerical answer. We were pleased that a significant number of candidates answered part (a) in Questions 1, 4 and 5 and then attempted to use the results in part (b).

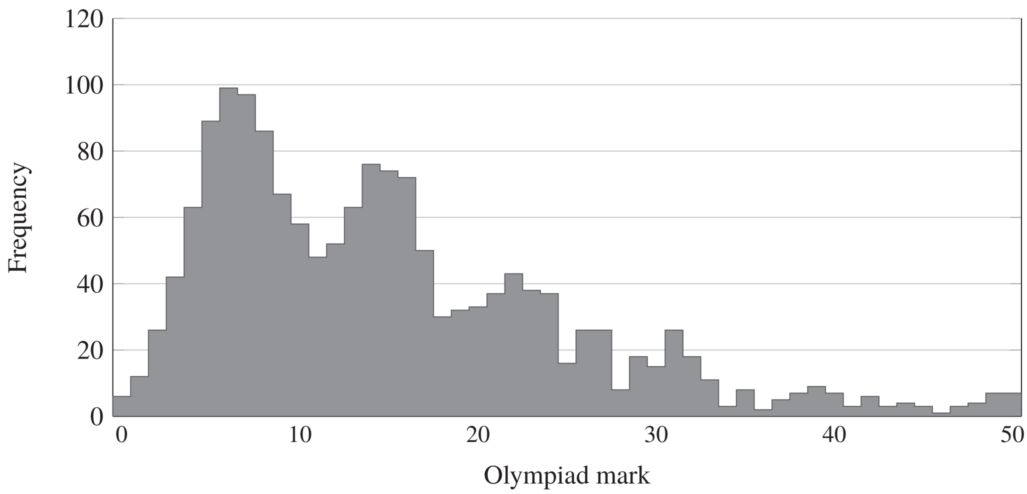
We saw many elegant and creative solutions, some of which we present as alternative solutions below. Many candidates demonstrated real mathematical potential, coming up with their own strategies to solve problems in clearly unfamiliar areas. There were candidates who achieved a high mark for the whole paper and, even more pleasingly, many who produced excellent solutions to individual questions.

One of the most common mistakes was making unjustified assumptions, sometimes inadvertently assuming the thing we are trying to prove. This was most apparent in Question 2, where some candidates used properties of isosceles triangles before proving that the triangle in question is indeed isosceles, and in Question 5 part (a), where many made claims such as 'the remainder of a square number is always a square'.

Candidates are often unsure what they can assume and what needs to be proved. Some spend too much time explaining well-known results, while others lose marks for not justifying more complicated claims. A useful guide may be thinking about trying to explain your solution to a class mate.

The 2015 Mathematical Olympiad for Girls attracted 1577 entries. The scripts were marked on 10th and 11th October in Cambridge by a team of Emily Bain, Natalie Behague, Magdalena Burrows, Andrew Carlotti, Andrea Chlebikova, Philip Coggins, James Cooper, James Cranch, Tim Cross, Richard Freeland, Adam Goucher, Maria Holdcroft, Daniel Hu, Andrew Jobbings, Vesna Kadelburg, Sam Maltby, Matei Mandache, David Mestel, Jessica Mulpas, Joseph Myers, Vicky Neale, Peter Neumann, Sylvia Neumann, Martin Orr, Roger Patterson, David Phillips, Eve Pound, Linden Ralph, Katya Richards, Kasia Warburton, Jerome Watson, and Joanna Yass.

Mark distribution



Question 1

(a) Expand and simplify $(a - b)(a^2 + ab + b^2)$.

(b) Find the value of
$$\frac{2016^3 + 2015^3}{2016^2 - 2015^2}.$$

Solution

(a) Expanding out the brackets and simplifying, we find that

$$\begin{aligned}(a - b)(a^2 + ab + b^2) &= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 \\ &= a^3 - b^3.\end{aligned}$$

(b) Let $m = 2016$ and $n = 2015$.

Using the factorisation

$$m^3 + n^3 = (m + n)(m^2 - mn + n^2),$$

and the difference of two squares $m^2 - n^2 = (m - n)(m + n)$, we may write the fraction as

$$\frac{(m + n)(m^2 - mn + n^2)}{(m - n)(m + n)} = \frac{m^2 - mn + n^2}{m - n}.$$

Since in our case $m - n = 1$, the value of this expression is

$$\begin{aligned}2016^2 - 2016 \times 2015 + 2015^2 &= 2016(2016 - 2015) + 2015^2 \\ &= 2016 + 2015^2 \\ &= 4\,062\,241.\end{aligned}$$

Alternative

(This solution does not use the result of part (a).)

Replace 2015 by n. Then we get

$$\begin{aligned}2016^3 &= (n + 1)^3 \\ &= n^3 + 3n^2 + 3n + 1,\end{aligned}$$

using the binomial expansion. Therefore the fraction is

$$\frac{(n^3 + 3n^2 + 3n + 1) + n^3}{(n^2 + 2n + 1) - n^2} = \frac{2n^3 + 3n^2 + 3n + 1}{2n + 1}.$$

Using algebraic division, we obtain

$$\begin{aligned}n^2 + n + 1 &= 2015^2 + 2015 + 1 \\ &= 4\,062\,241.\end{aligned}$$

Markers' comments

Part (a) was not meant to be demanding, but was intended to give an indication of *one* possible approach to the second part.

Candidates were usually successful, though a few were unable to collect like terms together after expanding the brackets.

It is well worth knowing the factorisation

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

as well as the result for 'a sum of two cubes', and results for other powers.

There are a large number of ways of tackling part (b). All correct methods were given credit, especially if the final answer was also correct.

Many candidates either used (a) directly, with $a = 2016$ and $b = -2015$, or used the related result $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ with $a = 2016$ and $b = 2015$.

Some chose to ignore (a) and used the binomial theorem to expand $(2015 + 1)^3$ or $(2016 - 1)^3$ (see the alternative solution given above). Those who did this and could manage the resulting division were often successful.

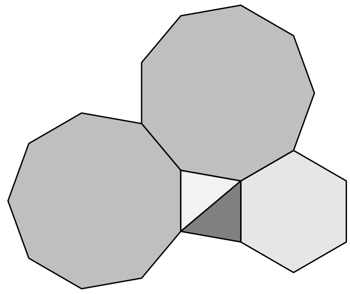
It was particularly pleasing to see scripts which used some means of simplifying the final calculation, for example, finding 2015×2016 to avoid squaring anything.

Rather too many candidates used brute force arithmetic from the outset—calculating 2015^3, for example—rather than looking for any "algebraic" method that might simplify the calculations. When some working was shown and the results were correct, this approach was rewarded with full marks, of course, but it is not recommended: apart from anything else, it is very easy to make a slip.

Question 2

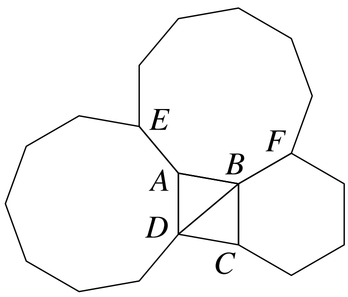
The diagram shows five polygons placed together edge-to-edge: two triangles, a regular hexagon and two regular nonagons.

Prove that each of the triangles is isosceles.



Solution

Label some of the vertices, as shown in the diagram.



First we show that triangle ABD is isosceles. Since the nonagons and the hexagon are regular and meet edge-to-edge, we have

$$DA = AE = AB.$$

This means that triangle ABD is isosceles.

We now look at triangle BCD. From the regular polygons, we find that

$$BC = BF = AB,$$

but we don't know anything about sides CD and BD. So let's calculate some angles.

Each angle in a regular hexagon is 120° and each angle in a regular nonagon is 140°. Therefore,

$$\begin{aligned}\angle DAB &= 360° - 140° - 140° \\ &= 80°\end{aligned}$$

and

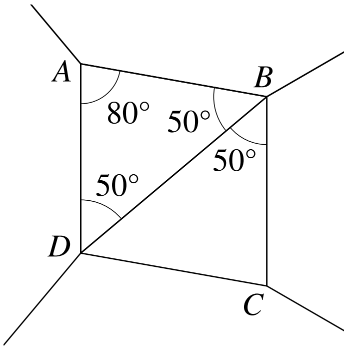
$$\begin{aligned}\angle ABC &= 360° - 140° - 120° \\ &= 100°.\end{aligned}$$

Since we already know that the triangle ABD is isosceles, we can calculate

$$\begin{aligned}\angle ABD &= \frac{180° - 80°}{2} \\ &= 50°\end{aligned}$$

and therefore

$$\begin{aligned}\angle DBC &= 100^\circ - 50^\circ \\ &= 50^\circ.\end{aligned}$$



We cannot directly find any other angles in triangle BCD. However, triangles BCD and BAD are congruent (SAS) because

$$\begin{aligned} BC &= AB, && \text{(from regular polygons)} \\ BD &= BD && \text{(common side)} \\ \text{and } \angle CBD &= \angle ABC. && \text{(each equals } 50^\circ\text{)} \end{aligned}$$

Since triangle BAD is isosceles, so is triangle BCD (with $BC = CD$).

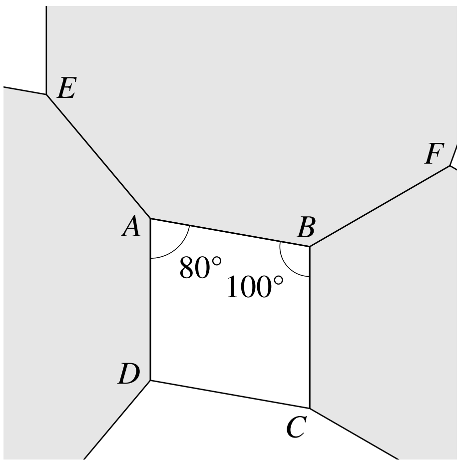
ALTERNATIVE

Each angle in a regular hexagon is 120° and each angle in a regular nonagon is 140°. Therefore,

$$\begin{aligned}\angle DAB &= 360^\circ - 140^\circ - 140^\circ \\ &= 80^\circ\end{aligned}$$

and

$$\begin{aligned}\angle ABC &= 360^\circ - 140^\circ - 120^\circ \\ &= 100^\circ.\end{aligned}$$



These two angles add up to 180° so, by the converse of internal angles on parallel lines, we find that AD and BC are parallel.

Since the nonagons and the hexagon are regular and meet edge-to-edge, we have

$$AD = EA = AB = BF = BC.$$

Hence AD and BC are equal and parallel. It follows that the quadrilateral $ABCD$ is a parallelogram. Therefore the sides AB and CD are equal.

But we already know that $AB = BC = DA$, so all four sides of $ABCD$ are in fact equal. Therefore the two required triangles are isosceles.

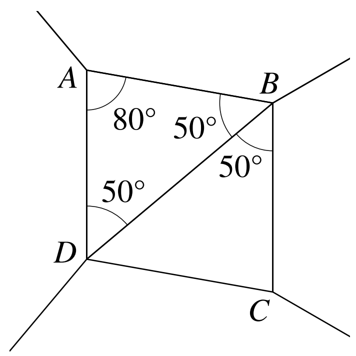
Alternative

We have already seen that $AB = AD$, so that triangle ABD is isosceles. Now $AB = BF = BC$ and so also triangle ABC is isosceles. Each angle in a regular hexagon is 120° and each angle in a regular nonagon is 140°. Therefore,

$$\begin{aligned}\angle DAB &= 360° - 140° - 140° \\ &= 80°, \\ \angle ABC &= 360° - 140° - 120° \\ &= 100°,\end{aligned}$$

and

$$\begin{aligned}\angle ABD &= \frac{180° - 80°}{2} \\ &= 50°.\end{aligned}$$



Thus BD bisects $\angle ABC$. Since triangle ABC is isosceles, BD is the perpendicular bisector of AC. Then also, since AC is perpendicular to BD and triangle ABD is isosceles, AC bisects BD. If X is the point where AC and BD meet then each triangle ABX, BCX, CDX, DAX is right-angled at X with one sides equal to AX, another equal to BX. Therefore their hypotenuses are equal (they are congruent), and $DC = BC$. Hence triangle BCD is isosceles.

Note

The quadrilateral $ABCD$ is in fact a rhombus, as noted by several candidates.

Markers' comments

This question saw many successful attempts, considerably more so than geometry questions in the past.

Quite a few candidates were concerned about checking that the triangles are not equilateral. Since every equilateral triangle is also isosceles, this was not necessary.

Many thought that they had to prove that base angles, as well as two sides, are equal. This sometimes resulted in loss of marks, when candidates used the fact that the triangle is isosceles without explicitly stating it.

Some candidates made an unjustified assumption that another congruent hexagon can be drawn using the side CD. But this assumes that CD is equal to the other three sides, which is what the question asked us to prove. This strategy sometimes went with an unsuccessful appeal to symmetry. Although E, A, C are collinear (since $\angle BAC$, as base angle of the isosceles triangle ABC, is $40°$), until the problem has been solved, the line EAC is a line of symmetry only of the figure consisting of the two nonagons and triangle ABD.

There were perhaps surprisingly few diagrams. Some students who did draw diagrams annotated them with the clause 'Not drawn to scale'. That makes no sense in a context like this—there is no relevant scale as the question is one of pure geometry. Although the diagram was given in the question, most candidates introduced their own labelling, and it would have been helpful to show this. Some candidates wisely chose to draw just relevant parts of the diagram in appropriate places in their solutions, and these were very helpful.

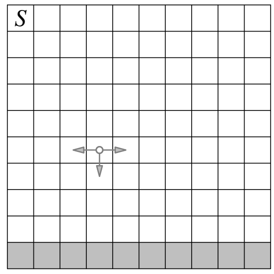
It was great to see many clear explanations of reasoning; we hope that candidates will find our solutions helpful for giving an idea of the level of detail we were looking for.

Question 3

A ladybird is going for a wander around a 10×10 board, subject to the following three rules (see the diagram).

(i) She starts in the top left cell, labelled S.
(ii) She only moves left, right or down, as indicated.
(iii) She never goes back to a cell that she has already visited.

In how many different ways can she reach the bottom row of cells, shaded grey?



SOLUTION

Once the ladybird (here called Ella) has reached the dth row she can move left or right, or not at all, to one of 10 cells in that row before her descent to the next row (unless she has already reached the last row).

Since Ella can't go back on herself, there is only one way for her to reach any of the ten cells in the dth row.

For each of rows $1, 2, \ldots, 9$, Ella has 10 options for the point at which she descends. Therefore there are 10^9, that is, $1\,000\,000\,000$, ways for Ella to reach the bottom row of cells.

NOTE

Wherever Ella starts in the top row, the answer is the same.

Also, if the board is of shape $m \times n$ (that is, m rows, n columns) then Ella can reach the bottom row from anywhere in the top row in n^{m-1} different ways.

MARKERS' COMMENTS

This question was the least popular, as an efficient strategy was difficult to identify. Those who did find it were usually successful in explaining why it worked.

The key observation is that, because of rule (iii), the ladybird has only one way of getting from one cell to another one on the same row. This means that we only need to count the number of choices for a cell from which to leave each row.

It was great to see some candidates starting with smaller boards, such as 2×2 or 3×3. This helped them to get a feel for the problem, and they were then able to predict the answer for the 10×10 board. Even where a candidate has spotted a pattern in this way, it is still important to explain *why* the pattern continues to hold for the larger board, by going back back to the problem and analysing the rules given.

It is important to be clear about exactly what is being counted, and to ensure that the multiplication makes sense. For example, if you are counting "the number of ways from the dth row to the next row", you might end up with a figure of 100 (for rows after the first), because there are 10 places to enter the row, 10 places to leave it, and exactly one path from entry to exit given those two places. This figure is correct, but it is not correct to produce an answer of 10×100^8 from it, because the ladybird must enter each row in the same column she left the previous row. Scripts that were too unclear about what was being counted tended to receive low marks.

Question 4

(a) A tournament has n contestants. Each contestant plays exactly one game against every other contestant. Explain why the total number of games is $\frac{1}{2}n(n-1)$.

(b) In a particular chess tournament, every contestant is supposed to play exactly one game against every other contestant. However, contestant A withdrew from the tournament after playing only ten games, and contestant B withdrew after just one game.

A total of 55 games were played.

Did A and B play each other?

Solution

(a) Contestants never play themselves but do play all the other $n-1$ competitors. Therefore there are $n(n-1)$ pairs (X, Y) such that X plays Y, with the contestants named in that order.

The game in which X plays Y is the same as that in which Y plays X however, so naming the contestants in order means that each game is counted twice. Therefore the total number of games is $\frac{1}{2}n(n-1)$.

(b) Suppose that there are m contestants other than A and B. The number of games those m contestants played with each other is $\frac{1}{2}m(m-1)$.

There are an additional 10 games played by A. If B did not play A then there is one additional game played by B, so the total number of games is

$$\frac{1}{2}m(m-1) + 10 + 1 = 55.$$

This simplifies to $m(m-1) = 88$, and this equation has no integer solutions, since $m(m-1) \leq 72$ when $m \leq 9$ and $m(m-1) \geq 90$ when $m \geq 10$.

Since we've found a contradiction by assuming they didn't play each other, we can be sure that A and B *did* play each other.

Note

It's not logically necessary, but we can analyse the situation where A and B did play as well. In this case there are no additional games played by B, so the equation for the total number of games played is simply

$$\frac{1}{2}m(m-1) + 10 = 55.$$

This simplifies to $m(m-1) = 90$, which has the solution $m = 10$. Hence, when we include A and B as well, there were 12 contestants at the start of the tournament.

Alternative

An alternative way of producing the formula for the number of matches is to mimic the proof of part (a), and try to count all participations. In the case where A and B did play each other, and as above there were m contestants not including A and B then there were:

- one contestant, A, who participated in ten games;

- one contestant, B, who participated in one game;
- nine contestants who each participated in m games (consisting of one game with A and $m-1$ games against the other unnamed contestants);
- $m-9$ contestants who participated in $m-1$ matches (one match against each of the other unnamed contestants).

Given that, by counting participations we are counting each match twice, the formula for the number of matches is

$$\frac{1}{2}(10+1+9m+(m-9)(m-1)) = \frac{1}{2}\left(m^2-m+20\right) = \frac{1}{2}m(m-1)+10$$

exactly as above.

The situation where A and B didn't play each other certainly can be analysed similarly, but is a little fiddlier. There are two cases: one where B's single opponent was also among A's opponents, and one where they weren't.

As a result, very few students who attempted this technique were completely successful.

Alternative

Another slick and clever method, which was spotted in full by a few students and in part by a few more, is to work out the number of contestants first, by exploring the cases with more than 12 and with fewer than 12 and showing they both produce contradictions.

Indeed, if there were 11 or fewer candidates, then at most 55 matches would be scheduled in the first place, which is impossible given that B dropped out after only one of them.

However, if there were 13 or more candidates, then the 11 or more who did not leave early would have played at least 55 matches between them, which is impossible given that A and B also played.

Hence there were exactly 12 candidates. The ten other than A and B played exactly 45 between them, leaving ten more. A played in all of these, and hence must have played B.

Markers' comments

There were many good answers to part (a), with clear reasoning carefully explained.

Some candidates tried to explain the meaning of each term in the formula, for example saying that $n-1$ is the number of games per contestant and $\frac{1}{2}n$ is the number of games going on at any time; this is not correct, because n might be odd whereas a number of games must be a whole number.

In part (b) it is important to realise what we need to prove. There are two possibilities: either A and B played each other or they didn't. Logically speaking, the most important thing is to find the one that's *impossible*; once we know that, we know that the other is the only possibility. (If we find that one of them is possible, on the other hand, that doesn't mean automatically that the other is impossible. Quite a lot of candidates concentrated on showing that it was possible for A and B to have played each other, but this is not sufficient to solve the problem.)

Candidates can't be expected to guess the right answer, so most scripts analysed both cases. There were many successful attempts, including some alternative solutions presented above.

Some candidates estimated that there must have been around 12 contestants and only analysed this special case. This does not provide a complete proof, unless an argument like that in the second alternative solution above is used.

Some candidates used the formula from (a) to create an equation like $\frac{1}{2}n(n-1) = 55$. But this formula only applies if everyone played everyone else, which was not the case here.

Question 5

(a) The integer N is a square. Find, with proof, all possible remainders when N is divided by 16.

(b) Find all positive integers m and n such that

$$m! + 76 = n^2.$$

[*The notation* $m!$ *stands for the* factorial *of* m, *that is,* $m! = m \times (m-1) \times \cdots \times 2 \times 1$. *For example,* $4! = 4 \times 3 \times 2 \times 1$.]

Solution

(a) Write $N = n^2$ and $n = 8k + r$ where r, the remainder when n is divided by 8, is one of 0, 1, 2, ..., 7. Then $N = (8k + r)^2 = 64k^2 + 16kr + r^2$, and this leaves the same remainder as does r^2 when divided by 16.

Those remainders R are shown in the following table.

r	R	r	R
0	0	4	0
1	1	5	9
2	4	6	4
3	9	7	1

Thus the possible remainders when N is divided by 16 are 0, 1, 4 and 9.

(b) If $m \geq 6$ then $m!$ is a multiple of $6! = 720$, which itself is a multiple of 16. Therefore the remainder when $m! + 76$ is divided by 16 is the same as the remainder when 76 is divided by 16, which is 12. From part (a) we know, then, that $m! + 76$ cannot be a square if $m \geq 6$.

If m is 1, 2 or 3, then the remainders after division of $m! + 76$ by 16 are 13, 14 and 2 respectively, and so $m! + 76$ cannot be a square.

But if $m = 4$ then $m! + 76 = 24 + 76 = 100 = 10^2$, and if $m = 5$ then $m! + 76 = 120 + 76 = 196 = 14^2$. Therefore the only pairs (m, n) of positive integers such that $m! + 76 = n^2$ are $(4, 10)$ and $(5, 14)$.

Note

We could look at remainders on division by some other numbers. For example, a similar argument works if we look at division by 7. For $m \geq 7$, $m! + 76$ gives remainder 6 when divided by 7, and we can show that a square number cannot give this remainder.

Markers' comments

There were some good solutions to the first part of the question. A lot of candidates found that 0, 1, 4 and 9 can all be obtained as remainders, but not all of them were careful to show that there are the *only* possibilities. Testing cases to find patterns can be a very helpful way of making progress in mathematics, but it then has to be backed up by careful analysis and understanding of the patterns in order to be sure that they continue.

There seemed to be some confusion about whether 0 counts as a remainder; we accepted both 'no remainder' and 'remainder 0' as correct answers. Some candidates gave remainders as fractions ($\frac{1}{16}, \frac{4}{16}, \frac{9}{16}$); this is not correct but was not penalised here.

Candidates often quoted "rules" they observed in this specific case, without realising that these rules did not hold in general. For example, we often saw the claim that 'the remainder of a square number is always a square number'. This is not true; for example dividing 16 by 9 gives remainder 7.

It was very pleasing to see so many candidates having the confidence to attempt the second part of final question. Many engaged with the question sufficiently to find both solutions, and a good number realised the importance of part (a) and tried looking at remainders.

Some candidates tried looking at division by 10, possibly because they have seen problems involving last digits of factorials. This approach does not work here, because 76 gives remainder 6 which is possible for a square number.

The crucial observation, that $m!$ is a multiple of 16 for $m \geq 6$, needed some justification. Once all but five possible cases are eliminated, we needed to see some evidence that these five cases were all checked.