United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

Tuesday 8 October 2019

Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. Full marks require clearly written solutions — not just answers — including complete proofs of any assertions you may make.
   Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.
   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Some questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on a fresh sheet of paper. Write on one side of the paper only.
   On each sheet of working write the number of the question in the top left-hand corner and your initials and the centre number (but NOT your name or school) in the top right-hand corner.
8. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
9. Staple all the pages neatly together in the top left hand corner.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Wednesday 9 October.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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1. At Mathsland Animal Shelter there are only cats and dogs. Unfortunately, one day 60 of the animals managed to escape. Once a volunteer had realised, they counted the remaining animals. They noted that half of the cats and a third of the dogs had escaped.

(a) (i) If the number of cats before the escape was C and the number of dogs before the escape was D, write down an equation linking C and D.

(ii) If the total number of animals before the escape was T, write down an equation linking C, D and T.

(4 marks)

(b) Given that more cats than dogs escaped, find the largest possible value of T. You must justify why the value you have found is the largest. (6 marks)

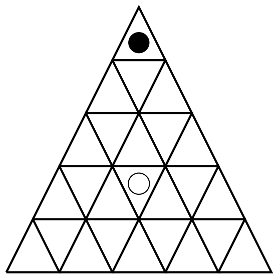
2. Beth has a black counter and Wendy has a white counter. Beth and Wendy move their counters on the two boards below according to the starting positions and rules given. They always move their counters at the same time.

(a) 

At each turn, each player moves their counter either one square to the left or one square to the right. Prove that the black and white counters can never be in the same square at the same time.

Hint You may find it helpful to refer to the colours of the squares on the board in your explanation. (3 marks)

(b) At each turn, each player moves their counter to a triangular cell which shares one edge with the cell that their counter is currently in. Can their counters ever be in the same cell at the same time? (7 marks)



Hint If you think the two counters can never be in the same cell at the same time, you should give an argument that they cannot be in the same cell at the same time which works no matter which sequence of moves Beth and Wendy do. If you think the two counters can be in the same cell at the same time, you should give an example of a sequence of moves after which they are in the same cell at the same time.

3. (a) Seth wants to know how many positive whole numbers from one to one hundred are divisible by two or five. He thinks that the answer is 70 because there are fifty multiples of two and twenty multiples of five from one to one hundred. Explain why his answer is too large. (2 marks)

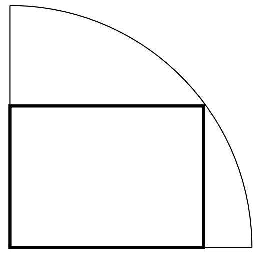
(b) Consider the list of 1800 fractions

$$\frac{1}{1800}, \frac{2}{1800}, \ldots, \frac{1799}{1800}, \frac{1800}{1800}.$$

How many are *not* in simplest form? Explain your reasoning. (8 marks)

[Note: The fraction $\frac{900}{1800}$ is not in simplest form because it can be simplified to $\frac{1}{2}$.]

4. The diagram shows a rectangle placed inside a quarter circle of radius 1, such that its vertices all lie on the perimeter of the quarter circle and one vertex coincides with the centre of the (whole) circle.



Let the perimeter of such a rectangle be P.

(a) Show that $P = 3$ is impossible. (4 marks)

(b) Find the largest possible value of P. You must fully justify why the value that you find is the largest. (4 marks)

Instead a rectangle is placed inside a whole circle of radius 1, such that its vertices all lie on the circumference of the circle.

(c) If the perimeter of the rectangle is as large as possible, show that the rectangle *must* be a square and calculate its perimeter. (2 marks)

5. Let n be a positive integer. Tracy writes a list of 10 whole numbers between 1 and n (inclusive). Each number in the list is either equal to, one less than, or one more than the number before it.

For example, when $n = 7$:

Her list could be 5, 5, 6, 7, 6, 6, 5, 5, 6, 6 or 4, 4, 3, 2, 1, 1, 1, 1, 1, 1.

Her list could *not* be 1, 3, 3, 4, 5, 5, 6, 7, 7, 7 or 5, 6, 7, 8, 7, 6, 5, 5, 5, 5.

(a) Suppose that $n = 3$. Stacey forms a list by copying Tracy's list, except that whenever Tracy writes a 1, Stacey writes a 3, and whenever Tracy writes a 3, Stacey writes a 1.

(i) Which lists could Tracy write that would cause her list to be the same as Stacey's?

(ii) Explain why Tracy can write as many lists that start 2, 2, 1 as start 2, 2, 3.

(3 marks)

(b) For which n between 1 and 10 (inclusive) is the number of lists that Tracy could write odd? (7 marks)

