United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

Thursday 7 October 2021

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Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. To gain full marks, your solution should be explained in full sentences. If your solution involves calculations, equations, tables, etc., explain where these come from and how you are using them. Explain how the steps of your solution link together, and give full proofs of assertions that you make. Answers alone will gain few marks (if any).
   Work in rough first, and then write up your best attempt at a clearly explained solution.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.
   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on an official master answer sheet that has a QR code on.
   On each additional answer sheet (blank or lined paper only) you need for a question please write your initials and the number of the question in the top left-hand corner.
8. Write on one side of the paper only as bold as possible.
9. Arrange your answer sheets in question order before they are collected. Please remove blank answer sheets i.e. those you do not wish to submit a solution for.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Friday 8 October.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121   challenges@ukmt.org.uk   www.ukmt.org.uk

1. (a) Find all whole numbers x such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = 1.$$

(7 marks)

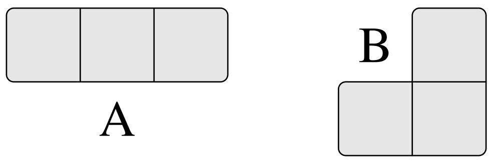
(b) Find all whole numbers x such that

$$(x^2 - 7x + 11)^{(x^2-4x+4)} = -1.$$

(3 marks)

2. Consider a 4×4 grid numbered 1 to 16 left to right then top to bottom. Tile A or Tile B is placed onto the grid so that it covers three adjacent numbers.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

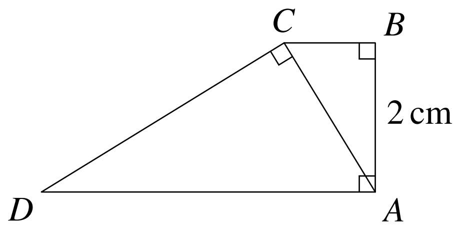


(a) If Tile A is placed onto the grid (the orientation of the tile may be changed), can the total of the **uncovered** numbers be a multiple of three? (3 marks)

(b) In how many different ways can Tile B be placed onto the grid (the orientation of the tile may be changed) so that the sum of the **uncovered** numbers is a multiple of three?

(7 marks)

3. The diagram shows a quadrilateral $ABCD$, where AB is 2 cm and $\angle ABC$, $\angle ACD$ and $\angle DAB$ are right angles.



(a) Let E be the point on DA such that CE is perpendicular to DA. Prove that triangles ABC and DEC are similar. (2 marks)

(b) Given that the area of quadrilateral $ABCD$ is 6 cm^2, find all possible values for the perimeter of quadrilateral $ABCD$. (8 marks)

4. Sam is playing a game. Her teacher gives her a positive whole number A, and then Sam chooses a positive whole number S. Sam then adds together all of the integers between S and $S + A - 1$ (inclusive) to obtain a total T. If T is even, Sam wins the game. For example, if $A = 4$, Sam can win by choosing $S = 10$ because then $T = 10 + 11 + 12 + 13 = 46$.

(a) (i) Show that if $A = 4$, Sam will win the game no matter which number she chooses.

(ii) Show that if A is a multiple of 4, Sam will win the game no matter which number she chooses. (3 marks)

(b) For which other values of A can Sam choose an S so that she wins? You must show how she can win for each of those values, and also explain why she cannot win for all the other values. (7 marks)

5. (a) By considering their difference, or otherwise, find all possibilities for the common factors of n and $n + 3$. (1 mark)

For $n \geq 2$, let $P(n)$ denote the largest prime factor of n.

(b) If a and b are positive integers greater than 1, explain why $P(ab)$ must be equal to at least one of $P(a)$ or $P(b)$. (1 mark)

(c) Find all positive integers n such that $P(n^2 + 2n + 1) = P(n^2 + 9n + 14)$. (8 marks)

