United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

supported by Overleaf

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions. They are not intended to be the 'best' possible solutions; in some cases we have suggested alternatives, but readers may come up with other equally good ideas.

All of the solutions include comments, which are intended to clarify the reasoning behind the selection of a particular method. You may want to try solving the problem yourself after reading the commentary. Full explanations are given even for the answer-only questions, to help you understand the ideas behind the solution.

Each question is marked out of 10. It is possible to have a lot of good ideas on a problem, and still score a small number of marks if they are not connected together well. On the other hand, if you've had all the necessary ideas to solve the problem, but made a calculation error or been unclear in your explanation, then you will normally receive nearly all the marks.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 365 1121 challenges@ukmt.org.uk www.ukmt.org.uk

1. The points A, B and C lie, in that order, on a straight line. Line CD is perpendicular to AC, and $CD = AB$. The point E lies on the line AD, between A and D, so that $EB = EC = AB$.

(a) Draw a diagram to show this information. Your diagram need not be accurate or to scale, but you should clearly indicate which lengths are equal. (2 marks)

(b) Calculate the size of the angle BAE. (8 marks)

Solution

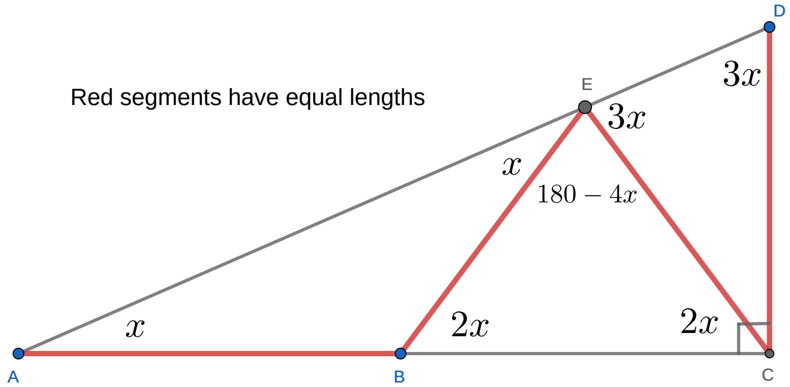
Commentary

Drawing a good diagram is an important part of solving any geometry question. Make sure you include all the information given in the question, but don't make any assumptions that are not explicitly stated. For example, drawing angle BAE as 45° would be misleading.

There are no angles given, so it seems impossible to do any angle calculations. However, you can label one of the angles and as unknown and try writing some equations. Usually it doesn't matter which angle you start from, but it seems sensible to call the angle you want to find x and then express as many angles as you can in terms of x. You can annotate your diagram as you go along, as shown in the solution below (the angle annotations are not needed to score marks for part (a)).

Another useful strategy in this type of question is to write down all the geometrical facts that are given in the question. This will both help you ensure that you have used all the available information, and also remind you that you need to give geometrical reasons for each step in your calculation. If you have drawn a nice diagram, you will see that in this question there are three isosceles triangles, two straight lines and one right angle. If you get stuck in your calculations, it is likely that you have not yet used one of those six facts. Once you have used all six facts, you should end up with an equation you can solve to find x.

(a) From the information in the question, we can draw the following diagram:



(b) Let $\angle BAE = x$. Then:

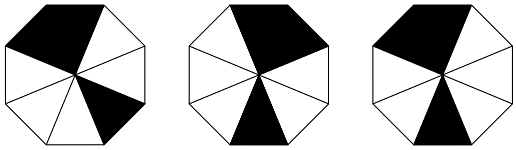
- Triangle ABE is isosceles, so $\angle AEB = x$ and $\angle ABE = 180 - 2x$.
- ABC is a straight line, so $\angle EBC = 2x$.
- Triangle EBC is isosceles, so $\angle ECB = 2x$ and $\angle BEC = 180 - 4x$.
- AED is a straight line, so $\angle DEC = 180 - x - (180 - 4x) = 3x$.
- Triangle DEC is isosceles, so $ECD = 3x$ and $\angle ECD = 180 - 6x$.
- We are given that ACD is a right angle, so

$$2x + (180 - 6x) = 90.$$

Solving this equation, we find that $\angle BAE = 22.5°$.

2. This question requires answers only.

In this question, two figures are considered to be different-looking if one cannot be rotated to produce the other. For example, in the diagram below, the first two figures are *not* different-looking, but the third one is different-looking from the first two.



(a) I have lots of congruent square tiles. Half of them are painted white and the other half are painted black. I fit four of these tiles together to make a larger square.

(i) Draw the two different-looking squares I can make using two white and two black tiles. (You can either colour your squares, or use the letters W and B to indicate colours.)

(ii) How many different-looking squares can I make in total?

(3 marks)

(b) I have lots of congruent tiles, each in the shape of an equilateral triangle. Half of them are painted white and the other half are painted black. I fit six of these tiles together to make a regular hexagon.

(i) How many different-looking hexagons can I make using three white and three black tiles?

(ii) How many different-looking hexagons can I make in total?

(7 marks)

Solution

Commentary

Although this question asked for answers only, it is important to ensure that the reasoning leading to your answer was correct, and think about how you would explain you reasoning to someone else.

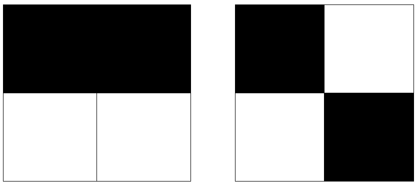
The biggest danger in counting questions is that you miss out some possibilities, or count some possibilities twice. It is therefore a good idea to think about different options first, before starting to count.

In both parts of this question, there are different cases depending of how many shapes of each colour are used. For example, in part (a) there are five cases: all black, three black and one white, two of each colour, one black and three white, or all white. Note that we have listed the cases systematically, according to the number of black squares, to make sure we don't miss out any.

It is also useful to ask whether you need to count each of those cases separately. For example, in part (a), the case of three black, one white and the case one black, three white are going to give the same number of possibilities; so you only need to count one of the cases and then double the answer. However, be careful: the 'two of each' case does not have a "partner", so this count should not be doubled. This observation will be even more useful in part (b), where there are more cases to consider.

Part (a)(i) is designed to get you thinking about when two figures are not different-looking (so should be counted only once). In part (b), the key insight is that you can rotate the hexagon to bring any triangle to any position you want. This means that you can fix the colour of one of the triangles, and then think about the number of ways to colour the others. Another good strategy is to think about the arrangement of "gaps" between the black triangles, as in the alternative solution.

(a) (i) Here are the two possible squares:



(ii) There are three different cases, depending how many triangles of each colour are used.

If only one colour is used, there are two possible squares (all white or all black). If there are three of one colour and one of the other, there is again only one possible square, so that's another two (3W+B or 3B+W). Finally, adding the two squares from part (i) gives the total of 6 different-looking squares.

(b) (i) Number the triangles 1 to 6, going clockwise. We can rotate the hexagon so that the triangle number 1 is black. We are going to place the other two black triangles moving clockwise.

If the second black triangle is number 2 then the third black triangle can be 3, 4 or 5. Note that the third black triangle can't be number 6, because then the hexagon could be rotated 60° clockwise so that the black triangles are $1, 2, 3$, which we have already counted. So there are three options with black triangles in positions 1 and 2.

If the second black triangle is in position 3, then the third one can only be in position 5. Putting it in position 6 would result in a hexagon that can be rotated 60° clockwise to produce one of the previous ones.

The second black triangles cannot be in position 5 because then the hexagon could be rotated by 120° so that positions 1 and 3 are black, but this case has already been covered. Similarly the second black triangle cannot be in position 6.

So there are four possible hexagons using three black and three white triangles.

(ii) There are four possible cases, depending on how many triangles of each colour are used: $3 + 3$, $2 + 4$, $1 + 5$ and $0 + 6$.

For the $3 + 3$ case, we found four possibilities in part (i). Note that each of the other three cases has two sub-cases, as black and white can be swapped (this was not the case for $3 + 3$). We are going to count the case where there are more white than black triangles, then double the total.

There is only one option for each of the $0 + 6$ and $1 + 5$ cases. For the case of two black and four white triangles we follow the similar strategy to part (i). Rotating the hexagon so that the triangle in position 1 is black, we can see that the second black triangle can be in positions 2, 3 or 4. If the second black triangle was in positions 5 or 6, the hexagon could be rotated to produce one of the previous arrangements. So there are three options with two black and four white triangles.

The total number of different-looking hexagons is therefore

$$4 + 2(1 + 1 + 3) = 14.$$

ALTERNATIVE

In part (b), for the $3 + 3$ and $2 + 4$ cases, we can think about the size of the gaps between the black triangles. If we record size of the gaps starting with the smallest and going clockwise, then each arrangement of the gaps will correspond to exactly one different-looking hexagon.

In the case of three black and three white triangles, there are three gaps, whose sizes should add up to 3. The four options are therefore $0 + 0 + 3$, $0 + 1 + 2$, $0 + 2 + 1$ and $1 + 1 + 1$.

In the case of two black and four white triangles, there are two gaps with sizes adding up to 4. The three options are $0 + 4$, $1 + 3$ and $2 + 2$.

We then combine those with the cases of one black and no black as in the previous solution.

3. This question requires answers only.

Define $f(x)$ to be the integer part of $\sqrt[3]{x}$; for example $\sqrt[3]{3.375} = 1.5$ so $f(3.375) = 1$, $\sqrt[3]{9} \approx 2.08$ so $f(9) = 2$, and $\sqrt[3]{27} = 3$ so $f(27) = 3$.

(a) Write down the first six positive cube numbers. Hence write down the value of $f(122)$.

(1 mark)

(b) If x is a positive integer with $f(x) = 3$, find the possible values of $f(2x)$. (3 marks)

(c) Find all positive integer values of x such that $f(x) + f(2x) + f(3x) = 10$. (6 marks)

Solution

Commentary

In any question involving an unfamiliar function or operation, it is a good idea to start by investigating what the function does, by looking at some examples. Hopefully you will soon notice that the value of $f(x)$ changes whenever x goes over a cube number.

It is possible to answer the final part of this question by making a table of values of x, $2x$ and $3x$ and the corresponding values of $f(x)$. However, part (b) encourages you to think about how $f(2x)$ is related to $f(x)$: If you know that $f(x) = 3$, what can you say about x, and therefore about $2x$?

For part (c), a useful hint is to think about how large x can be. You may want to think about how $f(2x)$ and $f(3x)$ compare to $f(x)$. You will need to consider different cases carefully to make sure you don't miss any solutions.

(a) The first six positive cube numbers are $1, 8, 27, 64, 125$ and 216. Since 122 is between 64 and 125, $f(122) = 4$.

(b) If $f(x) = 3$, we know that $27 \le x \le 63$. Then $54 \le 2x \le 126$, so $f(x)$ could be 3, 4 or 5.

(c) Notice that $\sqrt[3]{x} \le \sqrt[3]{2x} \le \sqrt[3]{3x}$, so $f(x)$ can be at most 3 (otherwise we would have at least $4 + 4 + 4$, which is greater than 10). Similarly, $f(3x)$ must be at least 4.

If $f(x) = 1$, then $1 \le x \le 7$ so $3x \le 21$ and $f(3x) \le 2$, which doesn't work.

If $f(x) = 2$, then $8 \le x \le 26$ so $3x \le 78$ and $f(3x) \le 4$. But then $f(2x)$ would need to be at least 4; however, $2x \le 52$ so this case is also impossible.

Finally, if $f(x) = 3$ we must have $f(2x) = 3$ and $f(3x) = 4$. This means that $27 \le x \le 63$, $27 \le 2x \le 63$ and $64 \le 3x \le 124$.

Solving the three inequalities, and remembering that x is an integer, gives:

$$27 \le x \le 63, \quad 14 \le x \le 31, \quad 22 \le x \le 41.$$

Combining the three inequalities together, the solution is $27 \le x \le 31$.

4. This question requires full written explanations.

Freya and Hilary play a game. Freya first chooses a positive integer a, with $1 \leq a \leq 2022$. Then Hilary chooses a positive integer b in response, with $1 \leq b \leq 2022$, where b may equal a.

Next they consider the sequence with nth term given by $an + b$ (for $n = 1, 2, 3\ldots$). If at least one term in the sequence is a multiple of ten then Freya wins the game and if not Hilary wins the game.

(a) Explain why, if Freya chooses $a = 2017$, Hilary cannot win the game. (1 mark)

(b) If Freya chooses $a = 2015$, for how many values of b will Hilary win the game? (2 marks)

(c) For how many values of a is it guaranteed that Freya will win the game, no matter Hilary's choice of b? (7 marks)

You should make it clear which values of a are included in your count, why Freya always wins for those values of a, and how Hilary can win for all other values of a.

Solution

Commentary

This question looks quite intimidating, and it takes some time to understand the rules of the game. Try selecting different values of a and b and writing out the first few terms in the sequence. You may want to start with the values suggested in parts (a) and (b). How many terms do you need to write down to determine who wins the game?

When writing your proof in this sort of question, you need to make sure that your argument is general. For example, if you are claiming that Hilary cannot win when Freya chooses a number that ends in a 7, your argument must apply to all numbers that end in a 7, not just one specific example. You also need to make it clear why Hilary loses for *all* possible choices of b.

On the other hand, if you are claiming that Hilary can win when Freya chooses a number that ends in a 5, you only need to give one example of a number Hilary can choose. So, for example, if you claim that Hilary can win when $a = 2015$, you only need to say 'If Hilary chooses $b = 1$, then all terms of the sequence end with 1 or 6 so Hilary wins'.

(a) If $a = 2017$ then, considering the values of n from 1 to 10, we can see that an ends in $7, 4, 1, 8, 5, 2, 9, 6, 3, 0$, i.e. it can end in any digit. This means that whatever value Hilary chooses for b, one of the first ten terms of the sequence will be a multiple of 10.

(b) If $a = 2015$ then an can only end in 5 or 0. Hilary will win if she chooses a value for b that does not end in 5 or 0. That is, she can choose any integer that is not a multiple of 5. Between 1 and 2022 there are $2020 \div 5 = 404$ multiples of 5, so Hilary can choose any of the remaining $2022 - 404 = 1618$ values of b.

(c) For Freya to guarantee a win, she must choose a value of a such that the sequence an contains numbers which end in every possible digit. Then, whatever Hilary chooses for b, one of the terms $an + b$ will end in a zero. Otherwise, if there is a digit that is not the ending of any of the an, Hilary will be able to choose a value for b that adds up to 10 with that digit.

Notice that we only needs to consider the last digit of a, and values of n from 1 to 10, because the last digits repeat after that. The table below shows all the possibilities:

a	$2a$	$3a$	$4a$	$5a$	$6a$	$7a$	$8a$	$9a$	$10a$
0	0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	9	0
2	4	6	8	0	2	4	6	8	0
3	6	9	2	5	8	1	4	7	0
4	8	2	6	0	4	8	2	6	0
5	0	5	0	5	0	5	0	5	0
6	2	8	4	0	6	2	8	4	0
7	4	1	8	5	2	9	6	3	0
8	6	4	2	0	8	6	4	2	0
9	8	7	6	5	4	3	2	1	0

We can see that Freya can choose any number that ends in 1, 3, 7 or 9. Each of those cases covers one-tenth of all the numbers from 1 to 2020, and then we need to include 2021 as well. Hence, the number of values Freya can choose from is $(2020 \div 10) \times 4 + 1 = 809$.

5. This question requires full written explanations.

(a) Given that m is a positive integer,

(i) What are the possibilities for the last digit of 2^m?

(ii) What are the possible remainders when 2^m is divided by 3?

(2 marks)

(b) Find all positive integer values of n, a and b, with $a \leq b$, such that $n! = 2^a + 2^b$. Justify carefully why there are no other possibilities.

(8 marks)

Note: For a positive integer n we define $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.

SOLUTION

COMMENTARY

Part (b) is a difficult problem, but part (a) gives us a useful hint: it suggests to look at the last digits and at divisibility by 3. You may have already noticed that large factorials are divisible by lots of numbers. In particular, all factorials from some point on are divisible by 3 and all factorials from some point on are divisible by 10. We can then split the problem into two parts: Investigating what happens for "large" values of n (when $n!$ is divisible by both 3 and 10; and separately checking the few "small" values of n. You have probably tried raising a number to consecutive powers and noticed that the last digit repeats. For example, for 2^m the pattern is $2, 4, 8, 6, 2, 4, 6, \ldots$. You may not know that this is true for remainders in general. In fact, last digit is just the remainder when a number is divided by 10. If you look at remainders on division of powers of 2 by 3 instead, you will find that the pattern is $2, 1, 2, 1, \ldots$. You may like to investigate why this is the case, and learn about *modular arithmetic* which develops rules for working with remainders.

(a) By raising 2 to consecutive positive integer powers, we find that both the last digits and the remainders on division by 3 form a repeating cycle. The possible last digits are $2, 4, 6$ and 8, and the possible remainders on division by 3 are 1 and 2.

(b) All factorials from 3! onwards are divisible by 3 and all factorials from 5! onwards end in a zero. So for $n \geq 5$, we need to find a and b such that $2^a + 2^b$ is divisible by both 3 and 10.

From the pattern of reminders found in part (a), for $2^a + 2^b$ to be a multiple of 3, one of a and b has to be even and the other on odd, as the remainders alternate between 1 and 2.

For $2^a + 2^b$ to end in a zero, there are two options: either one of 2^a and 2^b ends in 2 and the other one in 8, or one of them ends in 4 and the other one in 6. Since the pattern of the last digits is $2, 4, 8, 6, 2, 4, 8, 6, \ldots$ this means that a and b are either both even or both odd.

The conditions for divisibility by 3 and divisibility by 10 cannot be fulfilled at the same time, hence there are no solutions with $n \geq 5$.

It remains to check which one of the first four factorials can be written as a sum of powers of 2. The first four factorials are $1! = 1$, $2! = 2$, $3! = 6 = 2 + 4$ and $4! = 24 = 8 + 16$.

Hence the only solutions are $n = 3, a = 1, b = 2$ and $n = 4, a = 3, b = 4$.

ALTERNATIVE

It turns out that looking at divisibility by 7 gives a shorter way to solve this problem. The pattern of remainders when 2^m is divided by 7 is $2, 4, 1, 2, 4, 1, \ldots$, so the sum of two powers of 2 can never be divisible by 7. Since $n!$ is divisible by 7 for $n \geq 7$, the only possible solutions will have $n \leq 6$. It then only takes a bit of arithmetic to check that $5! = 120$ and $6! = 720$ cannot be written as sums of powers of 2.

ALTERNATIVE

When 2^m is divided by 15, it gives one of the remainders $1, 2, 4, 8$. There is no multiple of 15 being a sum of any two of those (in fact you need at least four), therefore no multiple of 15 is of the form $2^a + 2^b$. Since $n!$ is a multiple of 15 for $n \geq 5$, we only need to check $n \in \{1, 2, 3, 4\}$.

ALTERNATIVE

Rather than thinking about the sum of two powers of 2, we can re-write $2^a + 2^b$ as $2^a(1 + 2^{b-a})$. As noted in the previous alternative, $n!$ is divisible by 15 for $n \geq 5$. So we need $1 + 2^{b-a}$ to be divisible by 15 (since 2^a has no common factors with 15). But 2^m can only give remainders $1, 2, 4, 8$ when divided by 15, so this is not possible.

For $n \in \{1, 2, 3, 4\}$, we can find a by considering the highest power of 2 in $n!$: When $n = 1$, $a = 0$ (which is not possible), when $n = 2$ or 3, $a = 1$ and when $n = 4$, $a = 3$. The last two of those lead to the values of b we found before.

NOTE

One of the most well-known results of modular arithmetic is *Fermat's Little Theorem* and its generalisation using *Euler totient function*. They give the longest possible length of a cycle of remainders of powers.