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Mathematical Olympiad for Girls 2023

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Introduction

For many students, and possibly some teachers, this is the first experience of attempting a Maths Olympiad paper. It may therefore be useful to understand how these papers are marked, as students may be disappointed to receive a small number of marks for a problem they thought they had almost solved.

Most questions in Olympiad papers are marked using what we call the '0+/10−' principle. This means that the markers first read the whole write-up and decide whether the student has a viable strategy to solve the problem. It may be that there are some mistakes or small gaps in their reasoning, but if those could be relatively easily filled in then this response is marked in the '10−' regime, with usually up to three marks being taken away for gaps and mistakes. Common examples of small gaps are algebraic or arithmetical errors (provided they don't change the nature of the argument), missing one of several cases in a counting question, or lack of geometrical reasons when calculating angles.

If, on the other hand, the student has only started to explore the problem and has only made some useful observations, but does not have a strategy to generalise or prove them, then the script is marked in the '0+' regime. Up to three marks may be available for spotting a pattern or trying an idea which, if progressed further, could lead to a solution. Examples of this in the present paper would be, in Question 1(b), doing some angle calculations using the parallel lines or the isosceles triangle or, in Question 5(b), deriving results such as $S = 13 - c$ and $d = a + 1$. Notice that these all involve a substantial engagement with the problem, rather than just looking at some examples or special cases. Teachers should therefore reiterate to students that scoring even one or two marks on any of these questions is a real achievement.

The Mathematical Olympiad for Girls slightly different from other Maths Olympiads in that questions are broken down into several parts. Most of the time, the final part is the "main question" and the first part (or parts) are intended to suggest some useful results or good approaches to the problem. The reason for structuring the paper in this way is that the setters know that many of the candidates are not experienced in olympiad mathematics, and the hope is that by giving these pointers, we enable them to engage with the questions even if they are not familiar with standard olympiad techniques or "tricks". A useful hint is to read the whole question first and try to understand how the early parts may be helpful in solving the main problem.

The present paper also contains some answer-only questions, which nevertheless require a similar way of thinking to the usual olympiad questions. These questions are designed so that partial marks can be awarded to incorrect answers which show good progress towards the correct solution.

General comments

This is the second year that the paper has had the format of two answer-only questions and three full-solution questions, which seems to be proving popular, judging by yet another increase in the number of entries. Almost all candidates felt confident to tackle at least three different questions, and the markers were yet again impressed by a large number of high-quality write-ups.

The questions this year were based on common topics from the standard school syllabus, such as basic angle facts, the product rule for counting, inequalities, and algebraic proof using even and odd numbers. It was very pleasing to see so many students adapting their school knowledge to solve these more demanding problems.

While the knowledge required was relatively basic, careful thinking was needed in order to form a valid argument in all three full-solution questions. This was particularly true in Question 1, where candidates could do lots of useful angle calculations, but then had to select which ones were needed to answer the question. In Question 4, care was needed to ensure all cases were clearly covered and all the proposed solutions actually worked. Questions 2 and 3 obviously involved no written explanations, but it was still evident from the answers whether a candidate managed to make links between different parts of the question. Question 5 challenged even the strongest candidates, but it was good to see that many others managed to make some progress.

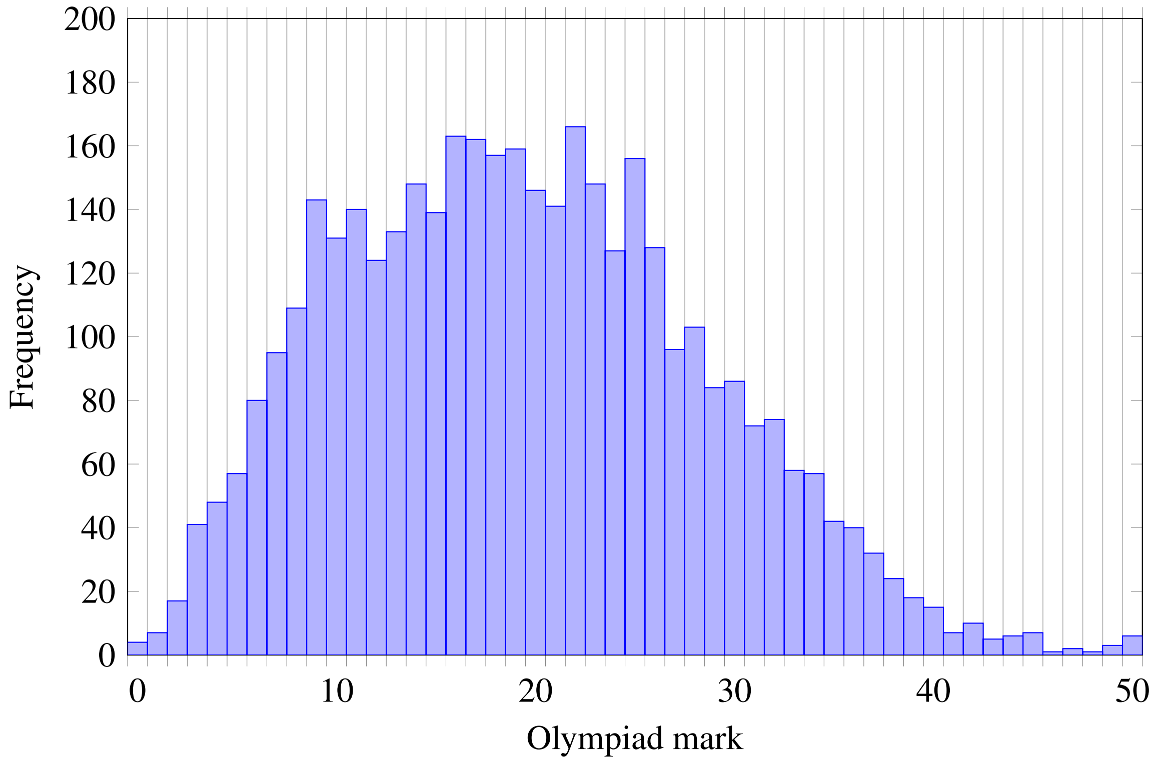
Overall, the standard of work seen was high and many of the full-response solutions were very well written. The engagement with the answer-only questions was particularly excellent.

It is unfortunately often the case that students think that they have solved a problem but only receive two or three marks. The most common reason for this is that their solution relies on a series of unjustified claims. There were two common examples in this paper. In Question 1, many candidates (correctly but without justification) claimed at the start that the quadrilateral $ABCD$ is a parallelogram, or maybe that $AB = CD$. In Question 5, it was common to see the claim that "c is maximal when $c = d$", which happens to be true in this case but has no justification in general. Students in this situation are strongly advised to read these comments and the official solution, to understand how they can add sufficient detail to their proofs.

Mark distribution

The MOG 2023 paper was marked by a team of Margaret Anthony, Richard Atkins, Ben Barker, Phillip Beckett, Lin Chen, Andrea Chlebikova, James Cranch, Stephen Darby, Gareth Davies, Wendy Dersley, Chris Eagle, Ben Fairfax, Richard Freeland, Chris Garton, Aleksander Goodier, Amit Goyal, Ben Handley, Jon Hart, Alexander Hurst, Michael Illing, Vesna Kadelburg, Jeremy King, Aleks Lishkov, Thomas Lowe, Linus Luu, Owen MacKenzie, Jack McMillan, Matei Mandache, Oliver Murray, Joseph Myers, Preeyan Parmar, Daniel Phillips, Christine Randall, Heerpal Sahota, Andjela Sarkovic, Amit Shah, Gurjot Singh, Alan Slomson, Geoff Smith, Marcus Smith, Rob Summerson, Stephen Tate, Jonathan Wallace, Harvey Yau and Dominic Yeo.

We received responses from 3918 candidates, an almost 30% increase on last year.



Question 1

This question requires full written explanations.

$ABCD$ is a quadrilateral, with vertices labelled in anti-clockwise order, such that:

AB is parallel to DC,

$AB = AC$, and

angle ADC is equal to angle ACB.

(a) Draw a diagram to show this information. Your diagram need not be to scale, but you should mark clearly equal lengths and angles.

(2 marks)

(b) (i) Prove that $AD = BC$ and that AD is parallel to BC.

(ii) What type of quadrilateral is $ABCD$? (8 marks)

Solution

(See the official solutions document)

Markers' comments

Geometry questions tend to split the field in every Olympiad, and it was no different here; 36% of the candidates essentially solved the problem, but 15% got no further than an attempt to draw a diagram. The mean mark was 4.9 and the median was 4.

Although the question required the knowledge of only basic geometrical facts, such as angles in isosceles triangles and on parallel lines, it was not easy to produce a coherent argument explaining how the given information leads to the required conclusions. Often candidates listed some facts they have derived and then some possible conclusions, without making it clear which follows from which.

For example, it was common to see arguments such as: "Angle DAB = angle DCB, angle ABC = angle ADB and AB is parallel to CD; therefore AD and BC are equal and parallel." This is in fact a correct deduction, but it is not clear which geometrical rules were used to draw the conclusion. In fact, the two angle equalities alone imply that the lines are parallel, but this needs to be either clearly quoted as a known result, or proved. The following argument is clear and would be acceptable: "Since angle DAB = angle DCB and angle ABC = angle ADB, the quadrilateral $ABCD$ has two pairs of equal opposite angles and therefore is a parallelogram; it follows that its opposite sides AD and BC are equal and parallel."

There were instances where a result was not quoted clearly and therefore the deduction made was not valid, even though the conclusion turns out to be true in this case. For example, it was quite common to see statements such as "Two lines which make equal angles with two parallel lines, must be parallel." This is not true in general: The configuration could be like in Figure 1 and not like in Figure 2.

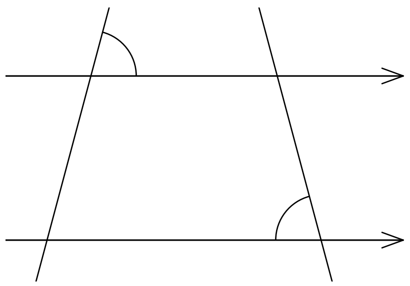


Figure 1

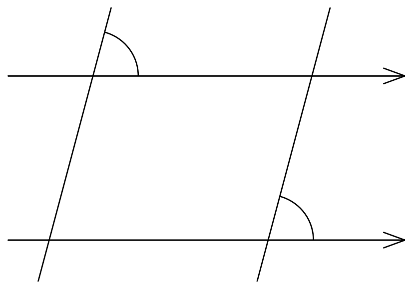


Figure 2

Another common mistake arose when using congruent triangles. Many students used alternate angles correctly, together with the given equal angles ADC and ACB (as shown in Figure 3) to conclude that the two triangles are similar. They then observed that they shared side AC and concluded that they must be congruent. However, Figure 4 shows that this is not necessarily the case, because the angles opposite the shared side AC may not be equal.

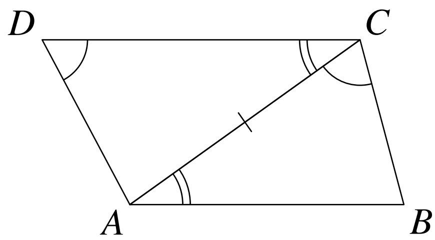


Figure 3

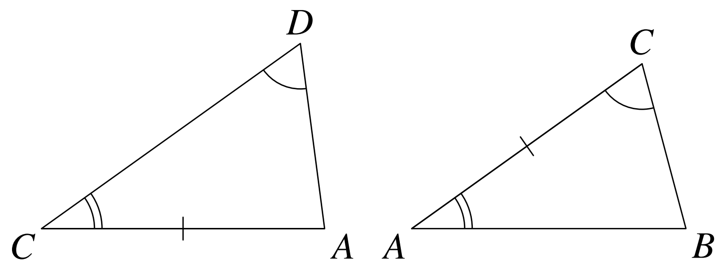


Figure 4

To conclude congruence, we need to point out either that angle ABC = angle ADB (as both are equal to angle ACB), or that $AB = AC$. This should serve as a reminder to students that, if they haven't used all the information given in the question (in this case, that $AB = AC$), then they probably haven't solved the problem.

Although many candidates showed confident use of angles of parallel lines (corresponding, alternate and co-interior), many also made mistakes. Two commonly seen errors are illustrated below. In Figure 5, alternate angles marked x are equal; for angles y and z to be equal, lines AD and BC would need to be parallel, which is not given at the start of the question. In Figure 6, angles marked x are equal corresponding angles, but angle y is not necessarily equal to them.

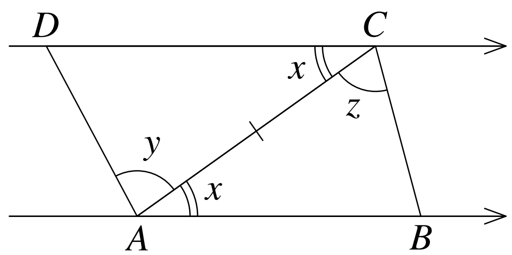


Figure 5

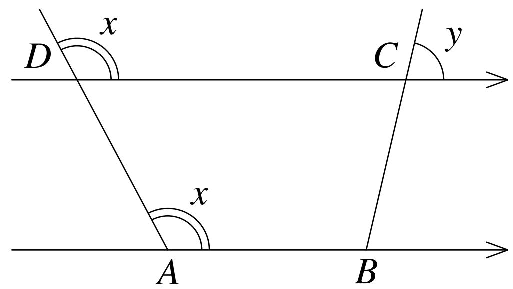


Figure 6

Some candidates anticipated that the shape was going to be a parallelogram and used its properties (such as opposite sides being equal) before they proved it. Some of the mistakes illustrated above possibly stemmed from this confusion, rather than misunderstanding of alternate and corresponding angles.

Despite all these mistakes, the vast majority of candidates approached the problem in a sensible way and made some useful observations, and the markers were impressed by the large number of clear and well explained solutions. This question does, however, illustrate the importance of paying attention to detail when structuring an argument and when using standard results.

Question 2

This question requires answers only.

In this question, $\overline{abc}$ denotes a three-digit number with digits a, b, c.

(a) Write down all three-digit multiples of 3 which only contain digits $1, 2$ and 3. Digits can be repeated.

(3 marks)

(b) (i) Write down the values of b for which $9b^b < 1000$.

(ii) Let a, b and c be non-zero digits. Find all three-digit numbers $\overline{abc}$ which satisfy the equation

$$3c^c + 6a^a + 9b^b = \overline{abc}.$$

(7 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

Almost all candidates attempted this question and around a half solved it, possibly with some minor errors. The mean mark was 6.3 and the median was 7.

Most candidates were able to list the required numbers in part (a), although it was common to see eight numbers instead of nine. The most commonly missed number was 222. In this problem, it was useful to remember that a number is a multiple of 3 if the sum of its digits is a multiple of 3.

Since the question specified that b was a digit, answers such as $b \leq 3$ were not accepted in part (b)(i). We did allow the inclusion of $b = 0$, although it should be noted that b^b is not defined in that case.

In the final part, some marks were available to candidates who made errors when checking and included some additional answers, as long as those were from their list of multiples of 3 in part (a).

Question 3

This question requires answers only.

(a) Five identical coins are placed in the cells of a 3×3 grid so that there is at most one coin in each cell and there is an odd number of coins in each row and each column.

(i) Show two examples of how this could be done.

(ii) In how many ways can this be done?

(3 marks)

(b) The numbers 1 to 9 are arranged in the cells of a 3×3 grid so that every row and every column have an odd sum.

(i) Show two examples of how this could be done.

(ii) How many such arrangements are possible?

You do not need to multiply out your answer, and may write it as a product, such as $2 \times 7 \times 289$ or 3×17.

(7 marks)

Solution

(See the official solutions document.)

Markers' comments

The markers were very pleased with the level of engagement with this question, with almost all candidates (over 90%) producing at least the four correct examples. Counting (combinatorics) forms a very small part of the school syllabus, and it was great to see the students taking this opportunity to explore an unfamiliar area of mathematics.

Over a third of the candidates solved the problem, possibly with one small error, and over a half of them scored full marks. Half of the rest produced partially correct answers to the counting parts of the question and were able to score some marks. The median mark was 4 and the mean was just under 5.

In part (a), the most common incorrect answers were 8 and 5, suggesting that students missed one of the three types of coin arrangements (the '+', the 'L', or the 'T'). Those who used an incorrect answer to part (a) correctly in part (b), were still able to score almost all the marks.

A pleasing number of candidates were able to make the link between the two parts of the question, as evidenced by their answer to (a) appearing in their calculation for (b). Most candidates heeded the advice of giving the final answer as a product, so arithmetical errors were rare.

Although this was an answer-only question, it was almost always possible to tell what mistake the candidate had made in part (b). Some common incorrect answers that scored relatively highly were:

- $4! \times 5!$ (This counts the number of arrangements with the odd numbers in some predetermined five cells.)

- $9 \times 5!$ (This ignores the different arrangements of even numbers for a set arrangement of odd numbers.)
- $9 \times (4! + 5!)$ (The numbers of arrangements of odd and even numbers are added instead of multiplied.)

Some other answers scored partial marks because it was clear that the candidates had the strategy of counting the number of arrangements of the odd numbers and the even numbers separately and then combining them in some way with the nine different options for where to place the odd numbers. For example, $9 \times 4^4 \times 5^5$ would be correct if the numbers were allowed to repeat.

Question 4

This question requires full written explanations.

A *Pythagorean triple* is a triple (x, y, z) of positive integers satisfying $x^2 + y^2 = z^2$. A triple is called *primitive* if no factor greater than 1 is shared by all of x, y and z.

(a) Show that, for every positive integer $n \geq 4$, $x = n^2 - 9$, $y = 6n$ and $z = n^2 + 9$ form a Pythagorean triple. Find one value of n for which this triple is *not* primitive.

(2 marks)

(b) Show that, in any Pythagorean triple, x and y cannot both be odd.

(2 marks)

(c) Find all primitive Pythagorean triples in which two of x, y and z differ by two.

Your solution must show that all the triples you found are primitive and that there are no other possibilities.

(6 marks)

You may not quote any general formulae for Pythagorean triples without proof.

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

It was pleasing to see that almost three-quarters of the candidates scored some marks on this question, even if only 180 made good progress in the final part (with 62 scoring 9 or 10).

The most common misinterpretation of the question was to assume that all Pythagorean triples have to have the form from part (a). The intention of this part was to illustrate the type of answer that was required in part (c) (i.e. infinitely many solutions expressed in terms of some parameter).

The second request in part (a) was checking the candidates' understanding of what "primitive" meant. It was therefore required to follow a calculation such as "when $n = 5$, we have $x = 16$, $y = 30$, $z = 34$" by a statement like "which is not primitive because all three are even".

Part (b) was done well, with many candidates showing awareness that an even square has to be a multiple to 4. A common mistake was to denote the two odd numbers by $2n + 1$ and $2n + 3$, or similar, which does not represent two general odd numbers.

Those who engaged fully with part (c) often managed to make substantial progress in the case $z = x + 2$, getting at least as far as $y^2 = 2\sqrt{x+1}$ and concluding that x has to be one less than a square number. Many also noted that y is even so x has to be odd. Candidates aiming for the top marks should be encouraged to express such statements algebraically: in this case that would mean writing $x = (2m)^2 - 1$, and then finding similar expressions for y and z.

Full marks required dealing with two details that some candidates missed:

- That $z = x + 2$ is the only possibility to consider, as x and y can be swapped, and $y = x + 2$ is impossible (they can't both be odd from part (b), and can't both be even because then the triple is not primitive).

- That all triples found (with $x = 4m^2 - 1$ etc.) are primitive. Most candidates considered whether the numbers are even or odd, but not whether they have any other common factors. (They don't, because x and z are consecutive odd numbers.) Note that it is sufficient to prove that x and z have no common factors, and then it doesn't matter whether y shares any common factors with x.

A small number of candidates clearly knew the general form for Pythagorean triples. The question stated that such results may not be used without proof, so those candidates were not able to score full marks.

Question 5

This question requires full written explanations.

(a) x, y and z are real numbers, with $x \leq y \leq z$ and

$$x + y = 4$$
$$y + z = 7$$

Let $T = x + y + z$.

(i) Show that $x \leq 2$ and that $T = 11 - y$.

(ii) Find the minimum possible value of T, giving one example of values of x, y and z where this occurs.

(2 marks)

(b) a, b, c, d and e are real numbers, with $a \leq b \leq c \leq d \leq e$ and

$$a + b + c = 4$$
$$b + c + d = 5$$
$$c + d + e = 9$$

Let $S = a + b + c + d + e$.

Find the minimum possible value of S, giving one example of values of a, b, c, d and e where this occurs.

Your solution must fully justify why no smaller value of S is possible.

(8 marks)

SOLUTION

(See the official solutions document.)

MARKERS' COMMENTS

This was a difficult problem, with only around 50 candidates finding a viable method to solve part (b), although almost a quarter of all candidates made good progress, and almost three quarters scored some marks.

A large number of candidates did not seem to know what 'real numbers' meant. They interpreted it as 'positive integers', which makes the question much easier (and does therefore not score any marks).

Part (a) is most efficiently answered using algebraic manipulation, such as: $y = 4 - x$, so $x \leq 4 - x$; hence $2x \leq 4$ and $x \leq 2$, and similarly for y. Some candidates were successful in using proof by contradiction here: If $x > 2$ then $y > 2$ (as it is greater or equal to x), so $x + y > 4$, which is not the case.

It was also possible to use proof by contradiction in part (b), once we have discovered the solution $a = 1, b = 1, c = 2, d = 2$ and $e = 5$. If $c > 2$, then $d > 2$ and so $a > 1$ (because

$d = a+1$) and so $b > 1$. Hence $a+b+c > 1+1+2 = 4$, which is impossible. Some candidates found this approach, but rarely expressed it that clearly.

The main difficulty in this question is the number of variables, inequalities and equations involved. Candidates were rarely successful unless they reduced the number of variables efficiently, usually to something like $S = 13 - c$ or $S = a + b + 9$. The former was more helpful since it involves only one variable, and tells us that we need to find the maximum possible value of c.

Inequalities are difficult to deal with, so it is important to have a sensible strategy. It is perilous to substitute inequalities into equations. It is much safer to substitute equations into inequalities, to reduce the number of variables.

It is not obvious which variables to eliminate, but we should reach a chain of inequalities such as $d - 1 \leq b \leq c \leq d \leq b + 4$. At this point, we may recognise that the right-hand inequality is redundant, since it follows from the left-hand one (with a bit to spare).

The more obvious approach is to use $b+c+d = 5$ to eliminate b, giving $d-1 \leq 5-c-d \leq c \leq d$. Most candidates tried to continue with algebraic manipulation, but only a small number were fully successful. A safer approach is to plot the three 2D inequalities $c + 2d \leq 6$, $2c + d \geq 5$ and $c \leq d$. The point in the "allowed region" with the greatest value of c is $(2, 2)$.

Many candidates claimed it was obvious that we need $a = b$ and $c = d$ for minimal S. This is far from obvious; why do we not also need $b = c$ and $d = e$ (the latter contradicting $a = b$, of course)?

Other candidates claimed that to minimise $S = a + b + 9$ we need to minimise a and b. That happens to work in this case, but it is not always true. Consider the constraints $a + 3b \leq 4$ and $a + 2b \leq 3$. We need the minimum possible $a = 1$, but we need the *maximum* possible $b = 1$.

