UK Maths Trust

Mathematical Olympiad for Girls

Wednesday 27 September 2023

Organised by the United Kingdom Mathematics Trust

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Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks.
4. Questions 2 and 3 require answers only. The spaces for answers are clearly indicated on the answer sheets.
5. Questions 1, 4 and 5 require full written explanations. If your solution involves calculations, equations, tables, etc., explain where these come from and how you are using them. Explain how the steps of your solution link together, and give full proofs of assertions that you make. Answers alone will gain few marks (if any).
6. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.
   However, one complete solution will gain more credit than several unfinished attempts.
7. Earlier questions tend to be easier. Questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
8. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
9. You may use rough paper to note down your ideas, but you should write up your solution on the answer sheet provided for each question.
10. Start each question on an official master answer sheet that has a QR code on it.
    You may use additional sheets (blank or lined paper only). On each additional sheet please write the number of the question in the top left-hand corner, followed by the QR code digits following the ':' symbol. Please do not write your name or initials on additional sheets.
11. Write on one side of the paper only.
12. Arrange your answer sheets in question order before they are collected. Please remove blank answer sheets.
13. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Saturday 30 September 2023.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

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www.ukmt.org.uk

1. This question requires full written explanations.

$ABCD$ is a quadrilateral, with vertices labelled in anti-clockwise order, such that:

AB is parallel to DC,

$AB = AC$, and

angle ADC is equal to angle ACB.

(a) Draw a diagram to show this information. Your diagram need not be to scale, but you should mark clearly equal lengths and angles.

(2 marks)

(b) (i) Prove that $AD = BC$ and that AD is parallel to BC.

(ii) What type of quadrilateral is $ABCD$? (8 marks)

2. This question requires answers only.

In this question, $\overline{abc}$ denotes a three-digit number with digits a, b, c.

(a) Write down all three-digit multiples of 3 which only contain digits $1, 2$ and 3. Digits can be repeated.

(3 marks)

(b) (i) Write down the values of b for which $9b^b < 1000$.

(ii) Let a, b and c be non-zero digits. Find all three-digit numbers $\overline{abc}$ which satisfy the equation

$$3c^c + 6a^a + 9b^b = \overline{abc}.$$

(7 marks)

3. This question requires answers only.

(a) Five identical coins are placed in the cells of a 3×3 grid so that there is at most one coin in each cell and there is an odd number of coins in each row and each column.

(i) Show two examples of how this could be done.

(ii) In how many ways can this be done?

(3 marks)

(b) The numbers 1 to 9 are arranged in the cells of a 3×3 grid so that every row and every column have an odd sum.

(i) Show two examples of how this could be done.

(ii) How many such arrangements are possible?

You do not need to multiply out your answer, and may write it as a product, such as $2 \times 7 \times 289$ or 3×17.

(7 marks)

4. This question requires full written explanations.

A *Pythagorean triple* is a triple (x, y, z) of positive integers satisfying $x^2 + y^2 = z^2$. A triple is called *primitive* if no factor greater than 1 is shared by all of x, y and z.

(a) Show that, for every positive integer $n \geq 4$, $x = n^2 - 9$, $y = 6n$ and $z = n^2 + 9$ form a Pythagorean triple. Find one value of n for which this triple is *not* primitive.

(2 marks)

(b) Show that, in any Pythagorean triple, x and y cannot both be odd.

(2 marks)

(c) Find all primitive Pythagorean triples in which two of x, y and z differ by two.

Your solution must show that all the triples you found are primitive and that there are no other possibilities.

(6 marks)

You may not quote any general formulae for Pythagorean triples without proof.

5. This question requires full written explanations.

(a) x, y and z are real numbers, with $x \leq y \leq z$ and

$$x + y = 4$$
$$y + z = 7$$

Let $T = x + y + z$.

(i) Show that $x \leq 2$ and that $T = 11 - y$.

(ii) Find the minimum possible value of T, giving one example of values of x, y and z where this occurs.

(2 marks)

(b) a, b, c, d and e are real numbers, with $a \leq b \leq c \leq d \leq e$ and

$$a + b + c = 4$$
$$b + c + d = 5$$
$$c + d + e = 9$$

Let $S = a + b + c + d + e$.

Find the minimum possible value of S, giving one example of values of a, b, c, d and e where this occurs.

Your solution must fully justify why no smaller value of S is possible.

(8 marks)

